Transverse Instability of Straight Vortex Lines in Dipolar Bose-Einstein Condensates

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The physics of vortex lines in dipolar condensates is studied. Because of the nonlocality of the dipolar interaction, the 3D character of the vortex plays a more important role in dipolar gases than in typical short-range interacting ones. In particular, the dipolar interaction significantly affects the stability of the transverse modes of the vortex line. Remarkably, in the presence of a periodic potential along the vortex line, the spectrum of transverse modes shows a rotonlike minimum, which eventually destabilizes the straight vortex when the BEC as a whole is still stable, opening the possibility for new scenarios for vortex-line configurations in dipolar gases.

Introduction.—When rotated at sufficiently large angular frequency, a superfluid develops vortex lines of zero density [1,2], around which the circulation is quantized due to the single-valued character of the corresponding wave function [3]. Quantized vortices constitute indeed one of the most important consequences of superfluidity, playing a fundamental role in various physical systems, as superconductors [4] and superfluid helium [5]. Because of the enormous progress in the control and manipulation of ultra cold gases, Bose-Einstein condensates (BECs) offer an enormously new physics in ultra cold gases [18–21]. Time-of-flight experiments in Chromium have allowed for the first observation of DDI effects in cold gases [22], which have been remarkably enhanced recently by means of Feshbach resonances [23].

Dipolar gases present a rich nonlinear physics, since the DDI leads to nonlocal nonlinearity, similar as that in plasmas [24] and nematic liquid crystals [25]. This nonlocal character has remarkable consequences for the physics of rotating dipolar gases. It has been shown that the critical angular frequency for vortex creation may be significantly affected by the DDI [26]. In addition, dipolar gases under fast rotation develop vortex lattices, which due to the DDI may be severely distorted [27], and even may change its configuration from the usual triangular Abrikosov lattice into other arrangements [28,29]. However, the previous analysis of vortices in dipolar BEC have been constrained to situations where the 3D character of the vortex is unimportant.

In this Letter, we analyze the effects of the DDI on the physics of vortex lines in dipolar BEC. Because of the long-range character of the DDI, different parts of the vortex line interact with each other, and hence the 3D character of the vortices plays a much more important role in dipolar gases than in usual short-range interacting ones. Remarkably, the DDI may significantly modify the vortex line stability. We show that under appropriate conditions, the dispersion law for transverse modes shows a rotonlike minimum, which for sufficiently large DDI may reach zero energy, destabilizing the Kelvin waves. This instability may trigger the so-called Donnelly-Glaberson (DG) known in the context of superfluid turbulence [30], but, interestingly, the Kelvin-wave instability may be introduced by the DDI itself, and not by any additional counterflow. We show that the Kelvin-wave instability

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may occur in situations in which the whole BEC is stable, opening the possibility for the observation of twisted vortex line configurations in dipolar BECs.

Effective model.—In the following, we consider a dipolar BEC of particles with mass \( m \) and electric dipole \( d \) (the results are equally valid for magnetic dipoles) oriented in the \( z \)-direction by a sufficiently large external field, and that hence interact via a dipole-dipole potential:

\[
V_d(r) = \alpha d^2(1 - 3\cos^2\theta)/r^3,
\]

where \( \theta \) is the angle formed by the vector joining the interacting particles and the dipole direction. The coefficient \( \alpha \) can be tuned within the range \(-1/2 \leq \alpha \leq 1\) by rotating the external field that orients the dipoles much faster than any other relevant time scale in the system [31]. At sufficiently low temperatures (and away from shape resonances [32]) the physics of the dipolar BEC is provided by a nonlocal nonlinear Schrödinger equation (NLSE) of the form

\[
\begin{align*}
\mathcal{L} & = \frac{\hbar}{2m} \frac{\partial^2\Phi(r)}{\partial t^2} + V_{\text{eff}}(z) + g|\Phi(r)|^2 \\
& \quad + \int d^2r'|\Phi(r')|^2 V_d(r - r') \Phi(r),
\end{align*}
\]

where \( g = 4\pi\hbar^2a/m \), with \( a > 0 \) the s-wave scattering length. The BEC is in a 1D optical lattice, \( V_{\text{eff}}(z) = \varepsilon E_R \sin^2(2\pi z/Q) \), where \( E_R = \hbar^2 Q^2/2m \) is the recoil energy and \( Q \) is the laser wave vector. In the tight-binding regime (sufficiently strong lattice) we can write \( \Phi(r, t) = \sum_j w(z - j)\psi_j(\vec{\rho}, t) \), where \( b = \pi/Q \), \( \vec{\rho} = (x, y) \), and \( w(z) \) is the Wannier function associated to the lowest energy band. Substituting this ansatz in Eq. (1) we obtain a discrete NLSE. We may then return to a continuous equation, where the presence of the lattice amounts for an effective mass along the lattice direction and for a renormalized coupling constant [33]:

\[
\begin{align*}
\mathcal{L} & = \frac{\hbar^2}{2m} \frac{\partial^2\Psi(r)}{\partial t^2} + V_{\text{eff}}(z) + g|\Psi(r)|^2 \\
& \quad + \int d^2r'|\Psi(r')|^2 V_d(r - r') \Psi(r),
\end{align*}
\]

where \( \Psi(r) = \psi_j(\vec{\rho})/\sqrt{b} \) is the coarse-grained wave function, \( \vec{\rho} = b g \int w(z)^4 dz + g_d\zeta \) [34], with \( g_d = 8\pi\hbar^2/3 \), \( m^* = \hbar^2/2m^* \) is the effective mass, and \( J = \int w(z) \times [-(\hbar^2/2m^*)\partial^2/\partial z^2 + V_{\text{eff}}(z)]w(z) + b) dz \). The validity of Eq. (2) is limited to \( |\mu| \) momenta \( k_\perp \ll 2\pi/b \), in which one can ignore the discreteness of lattice. In addition, note that the single-band model breaks down if the gap between the first and second band becomes comparable to other energy scales in the problem (\( m/m^* \rightarrow 1 \)).

Homogeneous solution.—We consider first an homogeneous solution \( \Psi_0(r, t) = \sqrt{n} \exp[-i\mu t/h] \), where \( n \) denotes the condensate density, and \( \mu = (g + \bar{V}_d(0))\bar{n} \) the chemical potential, with \( \bar{V}_d(\vec{k}) = g_d|3k_z^2/|\vec{k}|^2 - 1|/2 \). From the corresponding Bogoliubov–de Gennes (BdG) equations one gets that the energy \( \epsilon(\vec{k}) \) corresponding to an excitation of wave number \( \vec{k} \) fulfills:

\[
\epsilon(\vec{k}) = E_{\text{kin}}(\vec{k}) \times \left[ E_{\text{kin}}(\vec{k}) + E_{\text{int}}(\vec{k}) \right] \text{where } E_{\text{kin}}(\vec{k}) = \hbar^2 k^2 / 2m + \hbar^2 k_z^2 / 2m^*
\]

is the kinetic energy, and \( E_{\text{int}}(\vec{k}) = 2(g + \bar{V}_d(\vec{k}))\bar{n} \). Stable phonons (i.e., low-\( k \) excitations) are only possible if \( E_{\text{int}} > 0 \) for all directions, i.e., if \( \epsilon + \beta(3k_z^2/|\vec{k}|^2 - 1) > 0 \), with \( \beta = g_d/\bar{g} \). If \( g_d > 0 \) phonons with \( \vec{k} \) lying on the \( xy \) plane are unstable if \( \beta > 2 \), while for \( g_d < 0 \) phonons with \( \vec{k} \) along \( z \) are unstable if \( \beta < -1 \). Hence absolute phonon stability demands \(-1 < \beta < 2 \).

Vortex solution.—We consider at this point a condensate with a straight-vortex line along the \( z \) direction. The corresponding ground-state wave function is of the form:

\[
\Psi_0(r, t) = \phi_0(\rho) \exp[im(\varphi - \mu t/h)],\quad \varphi \text{ is the polar angle on the } xy \text{ plane.}
\]

The function \( \phi_0(\rho) \) fulfills

\[
\mu\phi_0(\rho) + \frac{\hbar^2}{2m^*} \left( -D_\rho + \frac{1}{\rho^2} \right) \phi_0(\rho) + \frac{g}{\rho^2} |\phi_0(\rho)|^2 \phi_0(\rho),
\]

where \( D_\rho = \frac{1}{\rho} \partial_\rho \rho \partial_\rho + \frac{\hbar}{g} = g + g_d/2 \). Note that, due to the homogeneity of \( \Psi_0 \) in the \( z \) direction, in Eq. (3) the DDI just regularizes the local term (a similar feature was observed in Ref. [19(a)]). The density of the vortex core is given by \( |\phi_0(\rho)|^2 \), which goes to zero at \( \rho = 0 \), and becomes equal to the bulk density \( \bar{n} \) at distances much larger than the corresponding healing length \( \xi = \hbar/(m^*\bar{n}^1/2) \). Note that due to the regularization of the contact interaction, the size \( \xi \) of the vortex core depends on the DDI. In particular, this dependence is exactly the opposite as that expected for 2D vortices, since in 2D similar arguments provide \( \bar{g} = g + g_d \). Hence, even for equal densities, the cores of 2D and 3D vortices can be remarkably different in a dipolar gas, differing significantly from the behavior of short-range interacting gases, where both 2D and 3D vortices would have the same core size.

The effects of the long-range character of the DDI become even more relevant in the physics of Kelvin modes. We consider excitations of the vortex line of the form:

\[
\Psi(r, t) = \Psi_0(r, t) + \chi(r, t) \exp[i(\varphi - \mu t/h)],
\]

where \( \chi(r, t) = \sum_{\mu}(u_i(\rho) \exp[i(\varphi l + qz - \epsilon l t/h)] - v_\mu(\rho) \exp[-i(\varphi l + qz - \epsilon t/h)]) \). Introducing this ansatz into (2) and linearizing in \( \chi \), one obtains the corresponding BdG equations

\[
eu_i(\rho) = \left[ \frac{\hbar^2}{2m^*} \left( -D_\rho + \frac{(l + 1)^2}{\rho^2} \frac{m^*}{m^*} q^2 \right) \right]
\]

where

\[
\begin{align*}
- \mu + \frac{2\bar{g}}{\bar{g}^*} |\psi_0(\rho)|^2 u_i(\rho) - \frac{3\beta}{2} \bar{g} q^2 \int_0^\infty \rho \rho' |\psi_0(\rho')|u_i(\rho') \rho \rho' \\
- v_\mu(\rho') F(q\rho, q\rho'),
\end{align*}
\]
where $F_i(x, x') = I_i(x_\gamma K_i(x_\gamma)$, where $I_i$ and $K_i$ are modified Bessel functions, and $x_\gamma = \max(x, x')$, $x_\delta = \min(x, x')$. For every $q$ we determine the lowest eigenvalue $\epsilon$, that provides the dispersion law discussed below. The first line at the right-hand side of Eqs. (4) and (5) is exactly the same as that expected for a vortex in a short-range interacting BEC [11], but with the regularized value $\bar{g}$. The last term at the right-hand side of both equations is directly linked to the long-range character of the DDI and, as we show below, leads to novel phenomena in the physics of Kelvin modes ($l = 1$) in dipolar BECs.

In absence of DDI [or equivalently from Eqs. (4) and (5) without the last integral term] the dispersion law at low momenta ($q \xi \ll 1$) is provided by the well-known expression 

$$
\epsilon(q) = -(\hbar^2 q^2/2m^*) \ln[(m/m^*)^{1/2} q \xi].
$$

The integral terms of Eqs. (4) and (5) significantly modify the Kelvin spectrum in a different way depending whether $\beta$ is positive or negative. In order to isolate the effect of the DDI on the core size with respect to the effect of the integral terms in Eqs. (4) and (5) we fix $\bar{g}$ and change the parameter $\beta$ which is proportional to the dipole-dipole coupling constant. Figure 1 shows that for increasing $\beta > 0$ the excitation energy clearly increases; i.e., the vortex line becomes stiffer against transverse modulations.

In order to obtain an intuitive picture of why this is so, one may sketch the vortex core as a 1D chain of dipolar holes. Dipolar holes interact in exactly the same way as dipolar particles, and hence maximally attract each other when aligned along the dipole direction, i.e., the $z$ direction. In this sense, the configuration of minimal dipolar energy is precisely that of a straight-vortex line along $z$. A wiggling of the line produces a displacement of the dipolar holes to the side, and hence an increase of the dipolar energy. As a consequence, the DDI leads to an enhanced stiffness of the vortex line. From this intuitively transparent picture, we can easily understand that exactly the opposite occurs when $\beta < 0$. In that case, the dipolar holes maximally repel each other when aligned along the $z$ direction. Hence, it is expected that the dipolar energy decreases when departing from the straight-vortex configuration. This results in a reduced energy of the excitations for $\beta < 0$ (Fig. 1), i.e., it becomes easier to wiggle the vortex line.

In principle, a sufficiently large DDI could destabilize the straight-vortex line. However, in the absence of an additional optical lattice along the $z$ direction, the destabilization would occur for values of $\beta < -1$, i.e., in the regime of phonon instability in which the whole dipolar BEC is unstable against local collapses. An increase of the potential depth of the additional lattice leads to a reduction of the role of the kinetic energy term $mq^2/m^*$ in Eqs. (4) and (5) that enhances the effect of the dipolar interaction on the dispersion law. As a consequence, as shown in Fig. 2, in addition to the $-q^2 \ln(q \xi)$ dependence at low $q$, a rotonlike minimum eventually appears at intermediate $q$. Note that, typically, the dispersion law presents a relatively abrupt change in its curvature for a given value of $q$ ($\approx 1.4$ in Fig. 2). Interestingly, this feature is given by an avoided level crossing of the lowest eigenvalues $\epsilon(q)$. As a consequence the character of the lowest Bogoliubov modes changes at this value of $q$. For a sufficiently small $(m/m^*)$, the roton minimum eventually reaches zero energy, and the straight-vortex line becomes unstable against a novel type of instability.

Figure 3 summarizes our results. As mentioned above, for $\beta < -1$ and $\beta > 2$ the whole BEC collapses due to phonon instability. For $0 < \beta < 2$, the DDI makes the vortex stiffer against Kelvin modes [i.e., $\epsilon(q)$ grows for all $q$, as shown for a particular case in Fig. 1]. For $-1 < \beta < 0$ the vortex is softer against transverse excitations [i.e., $\epsilon(q)$ decreases], and $\epsilon(q)$ develops a roton feature when $m/m^*$ decreases approaching the curve $(m/m^*)_c$ (thick curve in Fig. 3 obtained by calculating for each value of $\beta$ the value of $m/m^*$ for which $\epsilon(q)$ vanishes at least for some $q$). For $m/m^* < (m/m^*)_c$ the Kelvin waves become unstable.
Roton minima occur in superfluid Helium [1], and have been also predicted in trapped dipolar BECs [19]. In the latter case, roton instability leads to local collapses which destabilize the whole BEC [35] (although a trap may stabilize the gas leading to novel ground states [36]). We stress that although the roton feature in the Kelvin-wave spectrum is also induced by the DDI, its physics fundamentally differs from the roton-maxon feature discussed in Refs. [19,36]. The roton-like minimum occurs due to the long-range interaction between different positions in the vortex line, and the related instability is not linked to a destabilization of the whole gas, which is indeed stable when the Kelvin-wave instability appears, but to an instability of the vortex line against twisting that may trigger the DG instability [30].

**Summary.**—Because of the nonlocal interaction between different parts of the vortex line, the 3D character of the vortices is more crucial in dipolar BECs than in short-range interacting ones. Remarkably, the DDI significantly distorts the stability of the vortex line. In particular, under appropriate conditions (involving an additional 1D lattice), the Kelvin modes become unstable. This instability may trigger the DG instability known in superfluid turbulence, but it may be produced by the DDI itself and not by any additional counterflow. This intrinsically dipole-induced instability opens the possibility for novel vortex line configurations in dipolar BECs, which will be studied in forthcoming works [37].

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FIG. 3 (color online). Stable or unstable regimes for straight-vortex lines.

[32] Close to the shape resonances, the form of the pseudopotential must be in general corrected. See D. Wang, New J. Phys. 10, 053005 (2008), and references therein.
[34] $C \approx \sum_{j \neq 0} |\tilde{w}(2\pi j/b)|^2$, where $\tilde{f}$ is the Fourier transform of $w(z)$.
[37] We have observed vortex line twisting in direct 3D simulations of Eq. (2), whose detailed discussion is the subject of ongoing work.