Soliton-soliton scattering in dipolar Bose-Einstein condensates

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We analyze the scattering of bright solitons in dipolar Bose-Einstein condensates placed in unconnected layers. Whereas for short-range interactions unconnected layers are independent, a remarkable consequence of the dipole interaction is the appearance of nonlocal interlayer effects. In particular, we show that both for one- and two-dimensional solitons the interlayer interaction leads to an effective molecular potential between disconnected solitons, which induces a complex scattering physics between them, that includes inelastic fusion into soliton molecules, and strong inelastic resonances. In addition, contrary to the short-range interacting case, a two-dimensional soliton scattering is possible, in which inelastic spiraling occurs, resembling phenomena in photorefractive materials.

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I. INTRODUCTION

Up to very recently, typical experiments on ultracold gases involved particles interacting dominantly via a short-range isotropic potential, which, due to the very low energies involved, is fully determined by the corresponding s-wave scattering length. However, recent experiments on cold molecules [1], Rydberg atoms [2], and atoms with large magnetic moment [3], open a fascinating research area, namely that of dipolar gases, for which the dipole-dipole interaction (DDI) plays a significant or even dominant role. The DDI is long range and anisotropic (partially attractive), and leads to unusual physics in condensates [4–6], degenerated Fermi gases [7], and strongly correlated atomic systems [8]. It leads to the Einstein-de Haas effect in spinor condensates [9], and may be employed for quantum computation [10], and ultracold chemistry [11]. Recently, time-of-flight experiments in chromium have allowed for an observation of dipolar effects in quantum gases [12].

Interestingly, the physics of short-range interacting Bose-Einstein condensates (BECs) at low temperatures is given by a nonlinear Schrödinger equation (NLSE) with cubic local nonlinearity, similar to the one appearing in other systems, in particular in Kerr media in nonlinear optics. In one dimension, nonlinearity allows for solitonic solutions [13], which have been observed in BEC [14]. However, bright solitons are unstable in two dimensions and three dimensions. In periodic potentials multidimensional discrete solitons are possible [15], but they do not move in a multidimensional way, although the use of optical lattices has been proposed to move two dimensional (2D) and three dimensional (3D) discrete solitons along a free direction [16]. Other interesting possibility to stabilize high-dimensional solitons is to use Feshbach resonances to manage spatially and/or temporally the scattering length [17].

Due to the DDI, a dipolar BEC is described by a NLSE with nonlocal cubic nonlinearity [4–6], opening an interesting cross-disciplinary link between BEC and other nonlocal nonlinear media, as, e.g., plasmas [18], where the nonlocal response is induced by heating and ionization, and nematic liquid crystals, where it is the result of long-range molecular interactions [19]. Nonlocality plays a crucial role in the physics of solitons and modulation instability [20–22]. In particular, any symmetric nonlocal nonlinear response with positive definite Fourier spectrum has been mathematically shown to arrest collapse in arbitrary dimensions [21]. Indeed, multidimensional solitons have been experimentally observed in nematic liquid crystals [19]. Recently we showed that under realistic conditions, 2D solitons may be generated in dipolar BEC [23]. However, the anisotropic character of the DDI violates the conditions of Ref. [21], and as a consequence a stability window occurs, rather than a stability threshold for a sufficiently large dipole strength. In Ref. [23] we briefly studied the scattering of 2D dipolar solitons, which is inelastic [24], contrary to the one-dimensional (1D) solitons in local NLSE, due to the lack of integrability [25]. However, the analysis of the inelastic scattering was largely complicated by the spatial overlapping of the solitons.

In this paper, we consider dipolar BECs placed in disconnected layers (or wires in the 1D case) (Fig. 1). For the case of short-range interactions, the two layers are independent if the interlayer hopping is suppressed. A remarkable consequence of the DDI is that unconnected layers become coupled due to nonlocal density-density interactions, leading to interesting interlayer effects, as, e.g., the possibility of a BEC of filaments, recently discussed in Ref. [26]. In the following, we analyze the rich physics introduced by interlayer effects in the nonlinear properties of dipolar BECs. In particular, we show that this interlayer interaction leads to an

FIG. 1. (Color online) Schematic representation of the system considered.
effective molecular potential between fully disconnected solitons, allowing for a complex scattering physics between them. This physics includes inelastic fusion into excited soliton-molecules for sufficiently slow solitons, as well as strong inelastic resonances for intermediate velocities. In addition, we discuss a 2D soliton scattering scenario, which is not possible in short-range interacting condensates, showing that inelastic soliton spiraling similar to that observed in photorefractive materials [27,28] is possible in dipolar BEC. Finally, we consider the scattering of 1D dipolar solitons in unconnected wires, and comment about the observability in on-going experiments.

The paper is structured as follows. In the next section we present the physical system and the effective 2D model that we analyze. In Sec. III we briefly discuss the existence of 2D solitons in dipolar BEC, basically reviewing the results of Ref. [23]. In Sec. IV a variational method describing the dynamics of the soliton-soliton system is presented. In Sec. V we analyze the properties of a soliton molecule formed by two solitons. Section VI is devoted to the scattering of 2D solitons, whereas Sec. VII discusses the scattering of 1D solitons. Finally, we summarize our conclusions in Sec. VIII.

II. MODEL

In the following, we consider a dipolar BEC transversally confined in the z direction by a two-well potential, with wells located at \( z \pm z_0 \), and separated by a sufficiently large potential barrier which prevents tunneling between them. At each well the z confinement is approximated by a harmonic potential of frequency \( \omega_z \), whereas there is no confinement on the xy plane. We consider in each well a BEC of N particles with electric dipole \( d \) (the results are equally valid for magnetic dipoles) oriented in the \( z \) direction by a sufficiently large external field, and that hence interact via a dipole-dipole potential,

\[
V_d(\mathbf{r}) = g_d(1 - 3 \cos^2 \theta)/r^3,
\]

(1)

where \( g_d = \alpha N d^2/4 \pi \varepsilon_0 \), with \( \varepsilon_0 \) the vacuum permittivity, and \( \theta \) the angle formed by the vector joining the interacting particles and the dipole direction. The coefficient \( \alpha \) is a tunable parameter in the range \(-1/2 < \alpha < 1\). This tuning is achieved by rotating the external field that orients the dipoles much faster than any other relevant time scale in the system, as discussed in Ref. [29]. At sufficiently low temperatures, our system is described by the following two coupled NLSE with nonlocal nonlinearity [4]:

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \Psi_j(\mathbf{r}) = \left( -\frac{\hbar^2}{2 m} \nabla^2 + U_j(\mathbf{r}) + g|\Psi_j(\mathbf{r})|^2 \right) \frac{\partial}{\partial t} \Psi_j(\mathbf{r}) + \int d\mathbf{r}' V_d(\mathbf{r} - \mathbf{r}') \times \left[ |\Psi_1(\mathbf{r}')|^2 + |\Psi_{-1}(\mathbf{r}')|^2 \right] \Psi_j(\mathbf{r}),
\]

(2)

where \( j = \pm 1 \) is the layer index, \( \Psi_j \) are the wave functions at each well, \( U_j(\mathbf{r}) = m \omega_z^2 (z - jz_0)^2/2 \), \( f|\Psi_j(\mathbf{r}, t)|^2 d\mathbf{r} = 1 \), and \( g = 4 \hbar^2 / 3 m N \) characterizes the contact interaction with \( a \) the s-wave scattering length. In the following we consider \( a > 0 \), i.e., repulsive short-range interactions.

We assume a 2D model in each well. This approximation demands that the corresponding chemical potential \( \mu \ll \hbar \omega_z \). In that case, \( \Psi_j(\mathbf{r}) = \psi_j(\mathbf{r}) \varphi_j(z) \) with \( \varphi_j(z) \) the ground-state wave function of the harmonic oscillator in the layer \( j \). Employing this factorization, the convolution theorem, the Fourier transform of the dipole-dipole potential \( \tilde{V}_d(k) = g_d(3k_z^2/k^2 - 1) \), and integrating over the \( z \) direction, we arrive at a system of two coupled 2D NLSE,

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} \psi_j = \left( -\frac{\hbar^2}{2 m} \nabla^2 \varphi_j^2 + \frac{4 \sqrt{\pi} \mu_d}{3 \sqrt{2} L_z} \int \frac{d^2 \mathbf{r}_p}{(2\pi)^2} e^{i \mathbf{k} \cdot \mathbf{r}_p} \times \left[ \tilde{\psi}_j(k_z) \tilde{F}(k_z, 0) + \tilde{\psi}_j(k_z) \tilde{F}(k_z, 2\pi d L_z/l_j) \right] \right) \psi_j,
\]

(3)

where \( \tilde{F} = \hbar / m \omega_z \) is the harmonic-oscillator length, \( \tilde{\psi}_j \) is the Fourier transform of \( |\psi_j(\mathbf{r})|^2 \), and \( F(\sqrt{2}, \sqrt{2}) = 2e^{-\lambda^2} - (3\pi e x^2/2)(e^{-2\lambda^2}) \right) \) erf(x) is the complementary error function. Using Eq. (3), we study numerically the equilibrium properties and the dynamics of the unconnected 2D dipolar solitons.

III. TWO-DIMENSIONAL DIPOLE SOLITONS

In this section, we briefly review the main results obtained in Ref. [23] concerning the existence of stable 2D solitons in dipolar Bose-Einstein condensates. For this discussion we shall consider a single well centered at \( z = 0 \). We introduce a Gaussian Ansatz for the wave function

\[
\Psi_0(\mathbf{r}) = \frac{1}{\pi^{3/4} L_{p}^{3/2}} \exp \left( -\frac{x^2 + y^2}{2L_p^2} - \frac{z^2}{2L_z^2} \right),
\]

(4)

where \( L_z = \sqrt{\hbar / m \omega_z} \), and \( L_p \) and \( L_z \) are dimensionless variational parameters related with the widths in the xy plane and the z direction, respectively. We have numerically checked that this Ansatz is indeed a very good approximation of the numerical solutions of the corresponding Gross-Pitaevskii equation (GPE).

Using this Ansatz the energy of the system reads as follows:

\[
\frac{2E}{\hbar \omega_z} = \frac{1}{L_p^2} + \frac{1}{2L_z^2} + \frac{1}{2\pi^2 L_p L_z} \left[ \frac{\tilde{g}^2}{4\pi} + \frac{\tilde{g} d}{3} \right] \left( \frac{L_p}{L_z} \right) \left( \frac{L_z}{L_p} \right),
\]

(5)

where \( \tilde{g} = 2g / \hbar \omega_z L_z, \tilde{g}_d = 8\pi N a_{1j} L_z, \) and \( \tilde{g}_d = 2g_d / \hbar \omega_z L_z \), and

\[
f(\kappa) = (k^2 - 1)^{-1/2}(k^2 + 1 - 3k^2 H(\kappa)), \]

with \( H(\kappa) \) = \( \arctan(\kappa^{1/2}) \) / \( \sqrt{\kappa} - 1 \) \( 1+\sqrt{\kappa} \). A simplified picture may be achieved by considering, as we have assumed above, that the system is two dimensional, i.e., that the transversal confinement in \( z \) is strong enough to guarantee \( L_z = 1 \). In that case, both kinetic and interaction energy scale as \( 1/L_p^2 \). In the absence of dipole-dipole interactions (\( \tilde{g}_d = 0 \)), and irrespective of the value of \( L_p \), \( E(L_p) \) is, depending on the value of \( g \), monotonic either growing with \( L_p \) (collapse instability) or decreasing with \( L_p \) (expansion instability). This reflects the well-known fact that 2D solitons are not stable in NLSE with contact interactions. In the case
of a dipolar BEC, the situation is remarkably different, since the function $f$ depends explicitly on $L_g$. This allows for the appearance of a minimum in $E(L_g)$ (see Fig. 2), which from the asymptotic values of $f [f(0)=−1$ and $f(\kappa→∞)=2]$ should occur if

$$\frac{g_{\Delta}}{3\sqrt{2}\pi} < 1 + \frac{g}{2(2\pi)^{3/2}} < \frac{2g_{\Delta}}{3\sqrt{2}\pi}.$$  

(6)

A simple inspection shows that this condition can be fulfilled only if $g_{\Delta}<0$, i.e., only if the dipole is tuned as previously discussed with $\alpha<0$ (this is true also for $L_\perp \neq 1$). In that case, the tuning of the dipole-dipole interaction may allow for the observation of a stable soliton, characterized by an internal energy $E_\perp<0$ (see Fig. 2). Note that if $N\alpha/L_\perp \gg 1$, then we arrive at the condition $|\beta|>3/8\pi = 0.12$, where $\beta = \tilde{g}_{\Delta}/\tilde{g}$. A direct resolution of the corresponding 2D GPE for large $\tilde{g}$ shows stable solitons for $|\beta|>0.12$, in excellent agreement with the variational Ansatz.

In Ref. [23] we complemented this analysis with a study of the low-lying excitations of the solitons. The corresponding variational calculation confirmed that for sufficiently large dipole strength the 2D soliton may be stabilized. Interestingly, for a maximal dipole strength the 3D character of the system becomes crucial, leading to a different sort of instability, in this case against 3D collapse. In the following, we shall restrict ourselves to the two-dimensional regime (sufficiently low dipole). We shall extend the analysis of the 2D lowest-lying excitations to the more complicated case of two-coupled two-dimensional solitons in the two-well problem in Sec. IV.

IV. VARIATIONAL METHOD

We introduce a variational formalism which allows us to study the equilibrium properties and dynamics of the two solitons. We consider a Gaussian Ansatz [30],

$$\psi(\vec{r},t) = A \prod_{\eta=1}^{N_{\perp,\parallel}} \exp\left(-\frac{(\eta - j\eta_0)^2}{4w_{\eta}^2}\right) \times \exp\left[i\frac{m}{\hbar} \left( j\eta_0 + (\eta - j\eta_0)^2 w_{\eta} \right)\right],$$

(7)

where $A$ is the normalization factor, $\{x_0,y_0\}$ is the soliton center, and $w_{x,y}$ the soliton widths. The variables $x_0,y_0,w_x,w_y$ are time dependent. The center-of-mass motion is an independent degree of freedom and it can be decoupled. Without loss of generality, it has not been included in the variational Ansatz. Introducing (7) into the corresponding Lagrangian [23], we obtain the following set of equations of motion:

$$m\ddot{q}_i = -\frac{\partial U}{\partial q_i},$$

(8)

where $q_{ij} = x_0,y_0,w_x,w_y$ are the dynamical variables. The problem then reduces to the analysis of an effective particle in a potential

$$U = \frac{\hbar^2}{8m} \left( \frac{1}{w_x^2} + \frac{1}{w_y^2} \right) + \frac{g}{\sqrt{2\pi} 8\pi w_x w_y L_z} + V,$$

(9)

that includes the dipolar interaction term

$$V = \frac{g_{\Delta}}{12\pi^2} \int dk \left( \frac{k^2}{k^2 + 1} \right) e^{-\frac{k^2}{2(k^2 + 1)}} \times [1 + \cos(2k_x x_0) \cos(2k_y y_0) \cos(2k_z z_0)],$$

(10)

that couples the unconnected solitons. This coupling gives rise to interesting interlayer effects which will be analyzed in the following sections.

V. STATIONARY SOLUTION OF THE TWO-COUPLED SOLITONS

A. Equilibrium position: Soliton molecule

Since the stabilization of the 2D solitons demands $g_{\Delta}<0$, the soliton-soliton potential is maximally repulsive for solitons on top of each other. However, due to the angular dependence of the dipole-dipole interaction, the potential becomes attractive at a given distance between the solitons, vanishing at long separations between them. As a consequence, the soliton-soliton potential presents a minimum (see Fig. 3), and hence the soliton pair can form a soliton molecule. The equilibrium separation between the solitons in the molecule (and the corresponding equilibrium widths of the solitons) can be obtained by calculating the minimum of po-
tential $U$. We have performed this task for different parameter regimes by means of a Powell-minimization procedure, and compared our results with those obtained from the direct numerical simulation of the coupled GPEs in imaginary time. Note that due to symmetry on the $xy$ plane, we can fix without loss of generality $y_0=0$. The equilibrium widths along the $y$ direction, i.e., the direction perpendicular to the line joining the solitons, are larger than that along the $x$ direction. The equilibrium position is by no means out of all that expected from approximating the solitons as pointlike objects. When the solitons approach each other, their structure becomes more relevant and that shifts the minimum of the molecular potential, but asymptotically both the potentials are the same. In the case of pointlike solitons, the interaction potential between the solitons would be provided by $V_{\text{point}} \propto -(x_0^2 + z_0^2)/(x_0^2 + z_0^2)^{5/2}$, which presents a minimum at $x_0 = 2z_0$. This value is certainly smaller than the results obtained from our variational or numerical calculations (see inset of Fig. 3).

Note that for growing $g$ the equilibrium separation between the solitons in the molecule is as expected reduced, becoming quasi-independent of $g$ for large $g$. This is explained since for large $g$ and fixed $\beta$ (i.e., increasing the number of atoms) the kinetic energy term becomes negligible compared to the interaction terms, and hence the minimum of $U$ just depends on $\beta$ and not on the particular values of $g$ and $g_d$.

B. Lowest-lying excitations

We estimate the frequencies of the lowest-lying excitations $\omega_j$ of the soliton molecule by means of our variational approach by calculating the eigenvalues of the Hessian of the potential $U$ at the equilibrium point. In order to address the excitation in which the soliton widths of one soliton can oscillate in phase and out of phase with the widths of the other soliton, we have extended our variational calculation to the case in which the soliton widths of both solitons are not necessarily the same. Hence, we consider five variational parameters $(x_0, w_{x1}, w_{y1}, w_{x2}, w_{y2})$. The frequency of the vibrational mode (in the inset of Fig. 4) associated with the dynamics of $x_0$ is much smaller than the monopole and quadrupole modes. In Fig. 4 we depict the frequencies of the internal modes (associated to the dynamics of the soliton widths) normalized by their values for independent solitons. A significant shift (as well as a splitting) of the frequencies of the internal modes is observed due to the presence of the second soliton. Additionally, the mode geometry is significantly modified, and whereas for independent solitons the modes have a perfect monopole and quadrupole symmetry, for interacting solitons the lowest energy mode becomes more pronounced along the $y$ direction. For values of $\beta$ approaching the instability threshold at which no stable soliton is possible (around $\beta = -0.15$ in Fig. 4), the internal modes of a single soliton transform in the presence of the second soliton into modes purely associated with oscillations of the $w_y$ and $w_z$ widths, respectively.

VI. SCATTERING OF TWO-DIMENSIONAL SOLITONS

In the following, we consider the scattering of 2D solitons. We first discuss the case where the relative velocity is parallel to the vector connecting the centers of mass of the two solitons ($v_{0z} = 0$). We shall denote this case as the 1D scattering scenario. We have studied the scattering for different initial relative velocities both by direct numerical simulations of Eqs. (3) and by determining the evolution of $\{x_0, w_x, w_y\}$ in our variational calculation. Figure 5 shows the variation of the soliton momentum as a function of the initial momentum. During the scattering process the total energy of the system is conserved but the coupling between the external degree of freedom (relative distance of the two centers of mass) and the internal degree of freedom (modes of the single solitons) leads to a dissipation in the dynamics of the external variables. It should be stressed that this dissipation is due to the extended structure of the dipolar solitons, and would be absent for pointlike solitons due to the absence of internal degrees of freedom.

As expected, for sufficiently large initial velocities the scattering may be considered as elastic. For sufficiently low velocities, the initial kinetic energy of the solitons is fully transformed during the inelastic scattering into internal soliton energy, and the initially independent solitons become

FIG. 4. Variational results for low-lying excitations: Solid lines are the breathing modes (in and out of phase) and dashed lines are the quadrupole modes (in-phase mode greater than out-of-phase mode for both monopole and quadrupole). The modes for the two solitons $\omega_{\text{two}}$ are normalized to the corresponding modes (of $\omega_{\text{one}}$) of an independent soliton [23]. The results are for $g/\sqrt{2\pi\hbar\omega l_z^2} = 200, z_0 = 3l_z$. Inset: The vibrational mode associated with the dynamics of $x_0$.

FIG. 5. Numerical (crosses) and variational (solid line) results for $\Delta k/k_0$ [$\Delta k = k_0 - k(t \rightarrow \infty)$] as a function of the initial momentum $k_0 l_z$ for $z_0 = 3l_z$, $g/\sqrt{2\pi\hbar\omega l_z^2} = 200, \beta = -0.2$. Inset: Numerical results with $z_0 = 4l_z$ (solid line) and $z_0 = 5l_z$ (dotted line).
For all the other cases (including the resonance discussed below) the two solitons have a relative momentum after the collision and the relative distance between the two centers of mass increases with time after the collision.

Interestingly, the inelastic losses do not increase monotonically for decreasing velocities, but on the contrary show a pronounced resonant peak at intermediate velocities (Fig. 5 and its inset). This effect is motivated by a coupling to internal soliton modes [32], which leads after the collision to a dramatic enhancement of the soliton widths that eventually may increase without limits leading to the destruction of the scattered solitons. We stress that this resonance behavior is only possible because internal modes of the 2D soliton are at rather low energies, well within the inelastic regime. For high momenta (right-hand part of the resonance) the two solitons cross each other, but the interaction time is not large enough to excite the internal modes of the solitons. For low momenta (left-hand part of the resonance) the relative kinetic energy is too small to allow the solitons to cross the potential barrier. They can either form a molecule or reflect off each other. If the interlayer distance is increased, the inelastic losses are as expected reduced, but an even more complicated structure of resonances is then resolved (Fig. 5, inset).

The possibility of generating stable 2D solitons in dipolar gases allows for a different scenario in soliton-soliton scattering that is not possible in short-range interacting condensates, that we denote as the 2D scattering scenario. Contrary to the case discussed above, in this scenario the relative velocity is not parallel to the vector connecting the centers of mass of the two solitons ($v_0 \neq 0$), leading to a nonvanishing angular momentum. This scattering scenario obviously demands stable 2D solitons, and hence it is not possible in short-range interacting condensates. The nonvanishing angular momentum is conserved after the collision, and as a consequence, for the case of soliton fusion at low velocities a spiraling motion is observed during the inelastic fusion. The solitons eventually stabilize into a rosettalike orbit around each other (Fig. 6). The spiraling motion of the solitons links the physics of dipolar BEC to that of photorefractive materials, where soliton spiraling has been proposed [27] and experimentally observed [28].

VII. ONE-DIMENSIONAL SOLITON SCATTERING

Finally, we consider 1D BECs placed at neighboring 1D sites. We consider two neighboring wires realized by means of a two-dimensional optical lattice (placed in the x-y plane), and the dipoles are oriented along the wire axis (z direction). At each wire, the transversal confinement is approximated by a harmonic potential $m\omega^2(x^2+y^2)/2$. Following similar scaling arguments as above, it is possible to show that for $g>0$, we obtain a stable bright soliton if $\beta > 3/4\pi$. Although, of course, some 2D features are missed in 1D, it is indeed possible to observe inelastic processes also in 1D solitons. This is particularly relevant for current experiments in $^{52}$Cr, since no tuning of the dipolar interaction is necessary for 1D solitons, easing very significantly the experimental requirements. For $^{52}$Cr a Feshbach resonance is necessary to satisfy the previous condition, but Feshbach resonances are well characterized and accessible [31].

As for the 2D case, we analyze the scattering of the coupled 1D solitons by means of a variational wave function,

$$\Psi(\vec{r},t) = A \exp \left( -\frac{(z-jz_0)^2}{2w_z^2} - \frac{(x-jx_0)^2+y^2}{2l_x^2} \right) \times \exp \left( \frac{i}{\hbar} \int \left( j(z-z_0)^2w_z^2 - \frac{w_z^2}{w_z} \right) \right),$$  

(11)

where $j=\pm 1$, $l_x = \sqrt{\hbar/m\omega}p$, is the transversal harmonic oscillator length, and $w_z$ and $z_0$ are variational parameters that describe the width of both solitons and their positions, respectively. After substituting the Ansatz in the corresponding Lagrangian, we obtain the following Euler-Lagrange equations:

$$m\ddot{z}_0 = -\frac{\partial}{\partial x_0} \mathcal{U}, \quad m\ddot{w}_z = -\frac{\partial}{\partial w_z} \mathcal{U},$$  

(12)

where again the problem reduces to an effective particle in a potential,

$$\mathcal{U} = \frac{\hbar^2}{8m w_z^2} + \frac{g}{2(2\pi)^3 w_z^2 r^2} + \mathcal{V},$$  

(13)

with $\mathcal{V}$ being the dipolar interaction

$$\mathcal{V} = \frac{8\ell}{12\pi^2} \int dk \left( \frac{k^2}{k_x^2} - 1 \right) e^{-k^2 l_x^2/2} e^{-k^2 l_y^2/2} e^{-k^2 l_z^2/2} \times [1 + \cos(2k_x x_0) \cos(2k_z z_0)].$$  

(14)

As for the 2D solitons, we have calculated the change of the soliton momentum as a function of the initial momentum, obtaining similar results as in 2D (Fig. 7), i.e., three main scattering regimes, soliton-fusion, resonant scattering, and elastic interaction. In Fig. 8 we depict an example of the dynamics of the soliton width in and out of resonance, which clearly shows a resonant (although nondestructive) behavior of the soliton widths for intermediate velocities.

VIII. CONCLUSIONS

Summarizing, interlayer effects are a fundamental feature introduced by the DDI in dipolar gases placed in uncon-
nected layers of an optical lattice. These effects may have remarkable consequences, as, e.g., the formation of a BEC of filaments [26]. In this paper we analyzed by means of numerical methods the rich physics introduced by interlayer effects in the nonlinear properties of dipolar BECs, and in particular in the scattering of unconnected solitons. The DDI induces an inelastic soliton-soliton scattering, that for low relative velocities, leads to the inelastic fusion into a soliton molecule. Interestingly, the inelastic losses do not increase monotonically for decreasing relative velocities, but on the contrary show strong resonances at intermediate velocities, at which, after interacting, the soliton widths are strongly modified, eventually leading to soliton destruction. This effect appears, because, due to the relatively low excitation frequencies of the solitons, a resonant coupling between incoming kinetic energy and internal soliton modes is possible for low relative velocities well within the inelastic regime. We have shown that a similar effect should be observable in 1D geometries, where the experimental requirements may be easily fulfilled in on-going chromium experiments. Finally, we have considered the 2D scattering of dipolar solitons, a unique possibility offered by the dipolar interactions in cold gases. We have shown that due to the combination of inelastic trapping and initial angular momentum a spiraling motion is possible, offering fascinating links to similar physics in photorefractive materials.

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[24] Strictly speaking, we should employ the term solitary wave. For simplicity we use the term soliton.


[32] However, a direct comparison between the internal modes discussed in Sec IV, and the position of the resonance just provides a qualitative understanding of the regime of velocities for which the resonance occurs. A more quantitative analysis is largely prevented, because the internal modes of the solitons vary dynamically when the solitons approach each other.