Quantum phase transitions in cold gases

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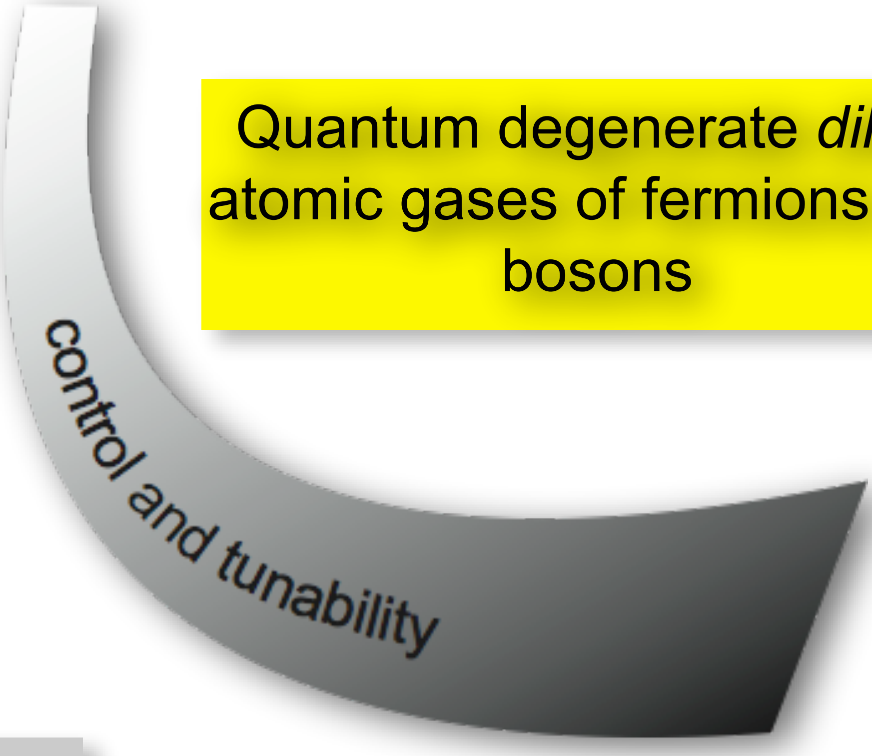
Atomic and molecular gases

Bose-Einstein condensation

- Gross-Pitaevskii equation
- non-linear dynamics



Quantum degenerate *dilute* atomic gases of fermions and bosons

Rotating condensates

- vortices
- fractional quantum Hall

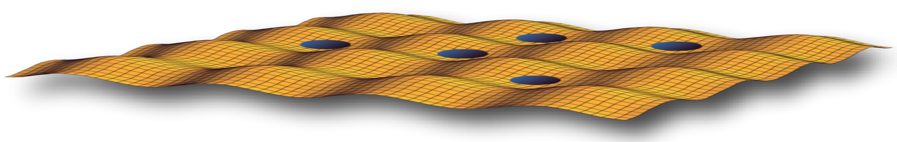
Molecules

- Feshbach resonances
- BCS-BEC crossover
- polar molecules

Optical lattices

- Hubbard models
- strong correlations
- exotic phases

Quantum gases in optical lattices



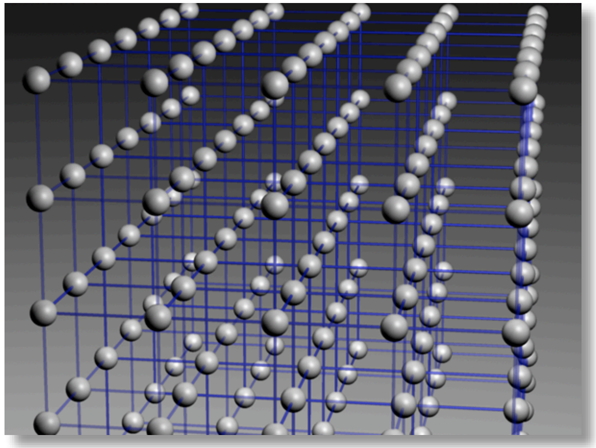
Experimental groups

- I. Bloch, Mainz
- T. Porto, NIST
- T. Esslinger, ETH Zürich
- D. Weiss, Penn State
- R. Grimm, Innsbruck
- W. Ketterle, MIT
- M. Inguscio, Florence
- J. Dalibard, Paris
- K. Sengstock, Hamburg
- M. Oberthaler
- M. Greiner, Harvard
- K. Zimmermann, Tübingen

Theory groups

- ...

Quantum gases in optical lattices

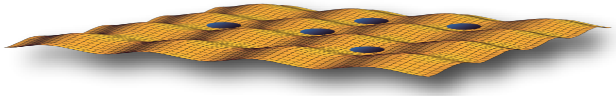


Optical lattices

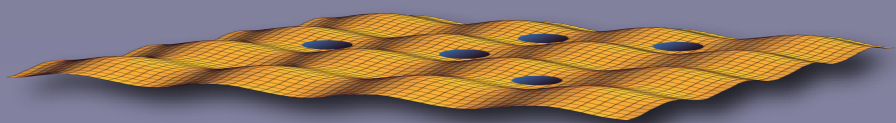
- properties of an optical lattice
- Bose-Hubbard model

Many-body theory

- Phase diagram
- Mean-field theory and effective theory
- 1D Bose-Hubbard model



Optical lattices

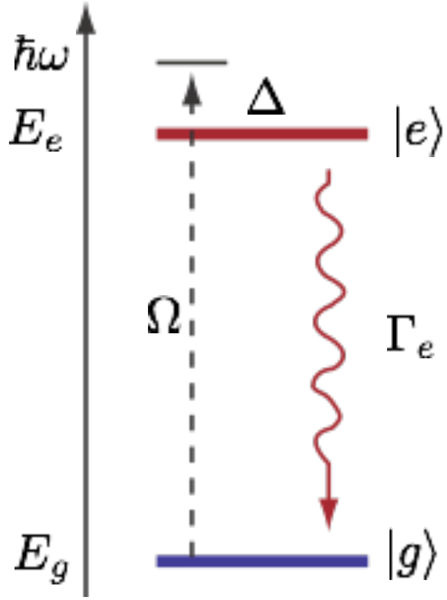


Interaction between light and atoms

- Hamiltonian between atoms and light:
  dipole approximation

$$H = -\mathbf{d}\mathbf{E}(t, \mathbf{r})$$

- external laser field: $\mathbf{E}(t) = \mathbf{E}_\omega e^{-i\omega t} + \mathbf{E}^*_\omega e^{i\omega t}$
- rabi frequency: $\Omega = |\langle e|\mathbf{d}\mathbf{E}_\omega|g\rangle|/\hbar$
- detuning: $\Delta = \omega - (E_e - E_g)/\hbar$



- AC Stark shift (change in the grounds
  state energy due to coupling to excited state)

$$\Delta E_g = -\alpha(\omega)|E_\omega|^2$$

$$\alpha(\omega) \approx \frac{|\langle e|\mathbf{d}\epsilon|g\rangle|^2}{E_e - E_g - \hbar\omega}$$

dynamical polarizability

blue detuned:

- low field seeker

red detuned:

- high field seeker

Interaction between light and atoms

- spontaneous emission: Γ_e

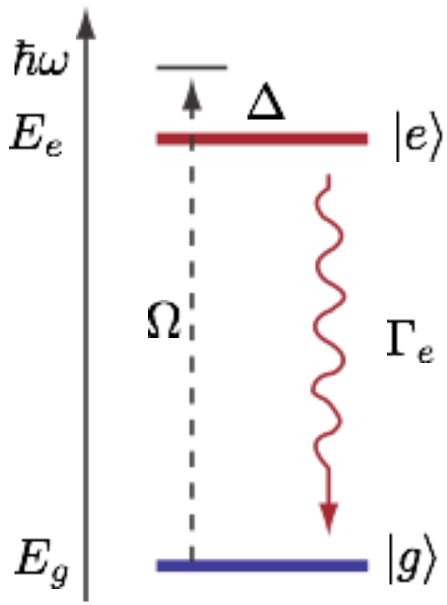
  excited state has a finite life time
  due to spontaneous emission

- AC Stark shift

$$\Delta E_g = \frac{\hbar\Omega^2\Delta}{\Delta^2 + \Gamma_e^2/4}$$

- loss of atoms from
  the ground state

$$\Gamma_g = \frac{\Omega^2\Gamma_e}{\Delta^2 + \Gamma_e^2/4}$$



- limits life-time of a BEC in an
  optical lattice

- requires large detuning

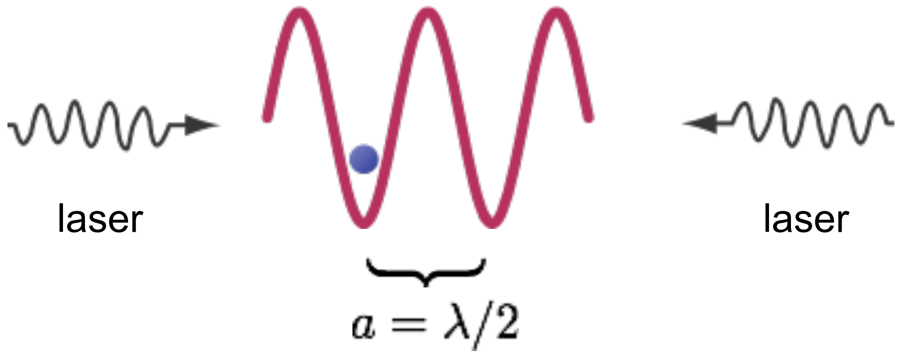
$$\Delta \gg \Gamma_e$$

- high laser power

Optical lattices

- a far-detuned standing laser wave provides
  a periodic potential for the particles

$$V(\mathbf{x}) = V_0 \cos(\mathbf{k}\mathbf{x})^2$$



- recoil energy: $E_r = \dfrac{\hbar^2 k^2}{2m}$

- structure in 3D

$$\mathbf{E}(t, \mathbf{r}) = \sum_i \mathbf{E}^i_{\omega_i} \cos(\mathbf{k}_i \mathbf{r}) e^{-i\omega_i t} + c.c.$$

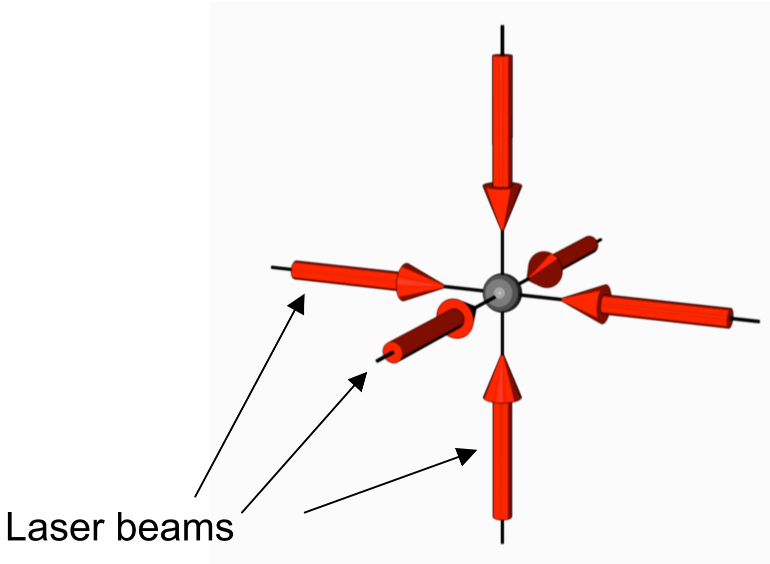
$\mathbf{k}_i$: wave length fixed by
the atomic transition

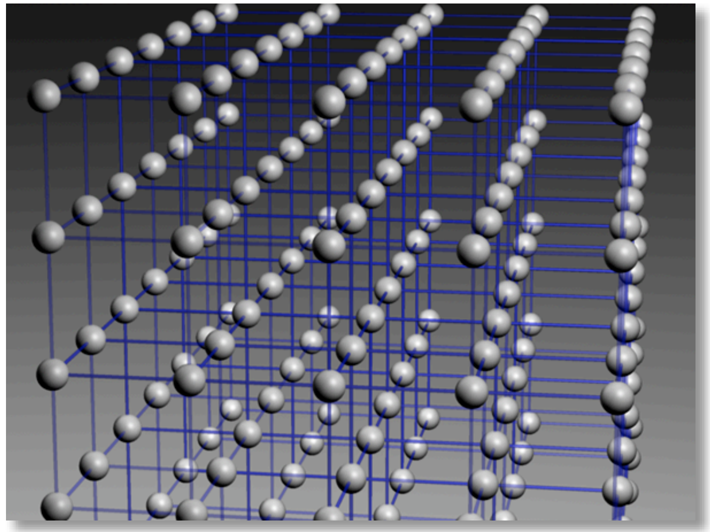
ω_i : slightly different frequencies
to cancel cross terms

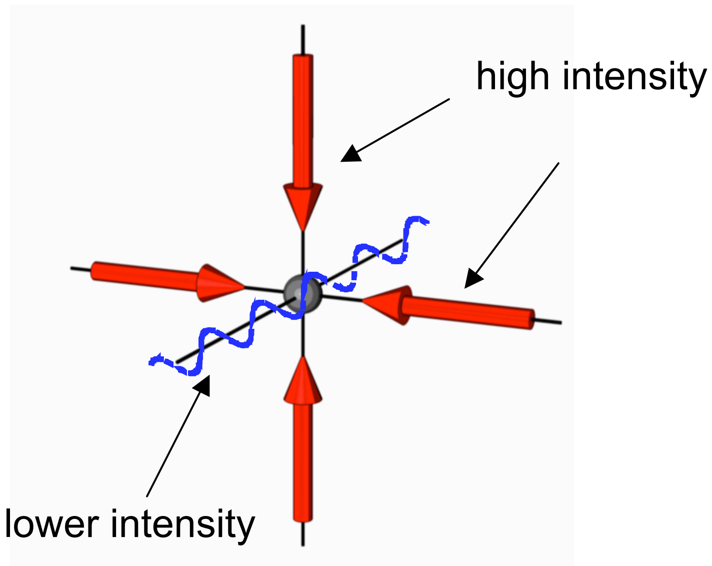
$\mathbf{E}^i_{\omega}$: polarization as additional
degree of freedom

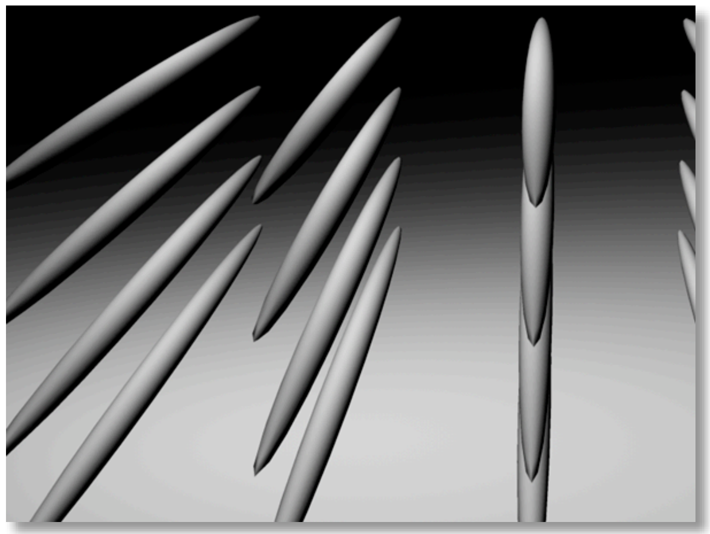
$$\Rightarrow \quad V(\mathbf{x}) = \sum_i V_i \cos(\mathbf{k}_i \mathbf{r})^2$$

Optical lattices





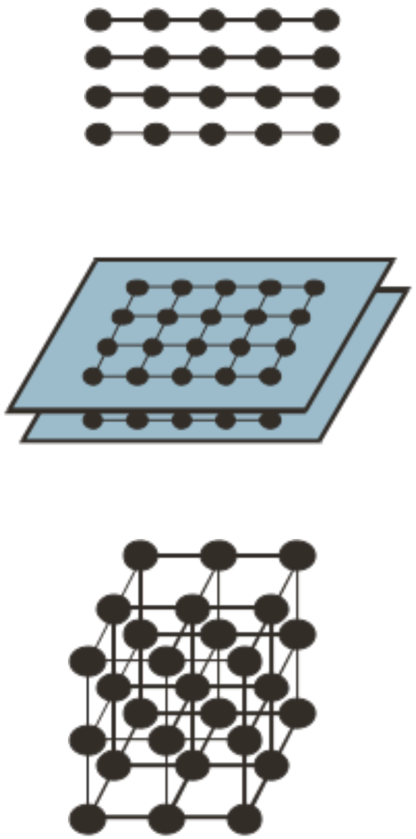




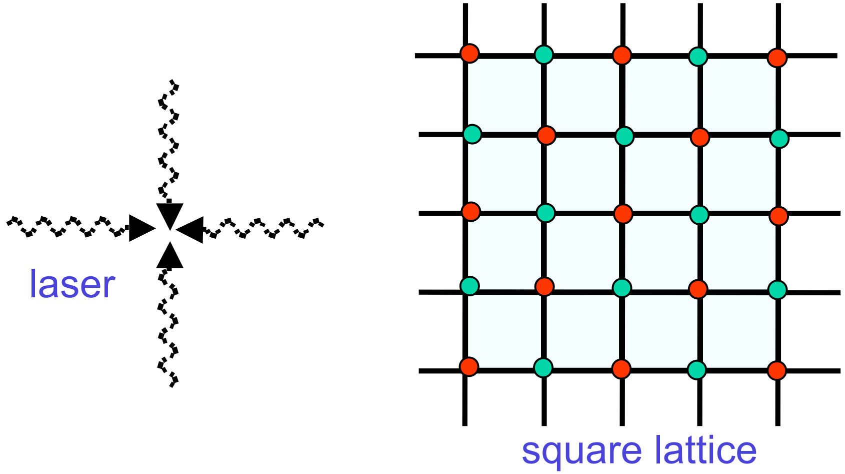
Optical lattices

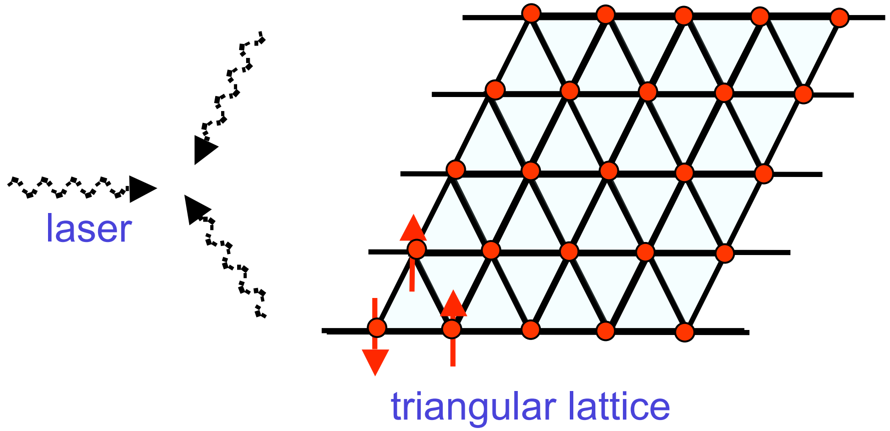
3D, 2D and 1D optical lattices

- dimensional crossover



Lattice geometries

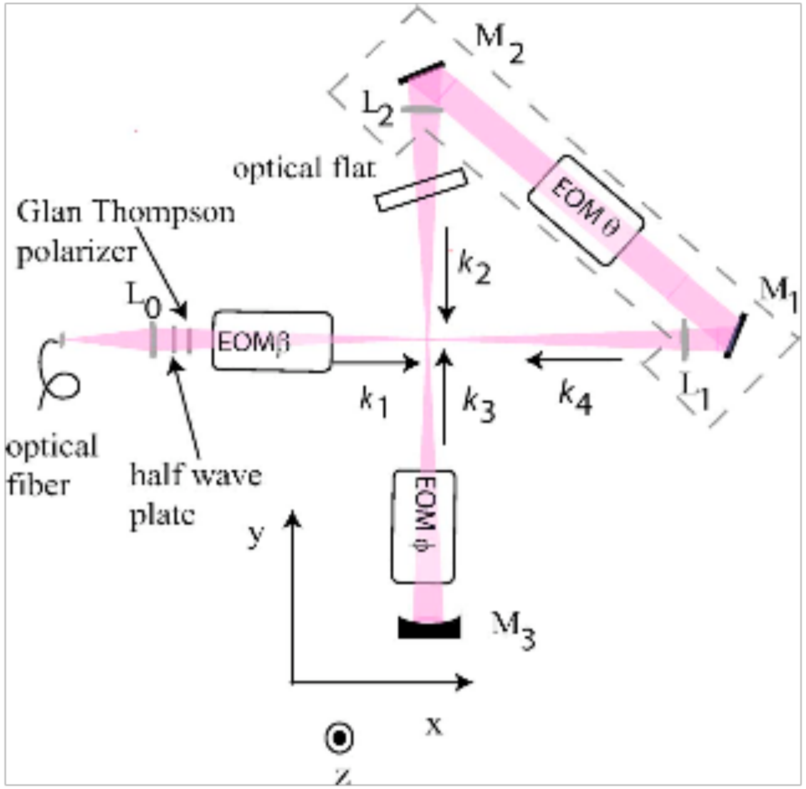


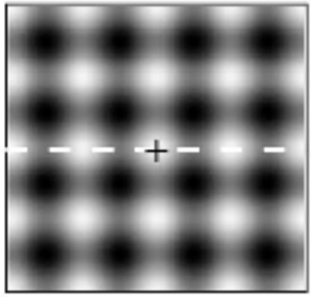


Optical lattices

Tricks with 2D optical lattice

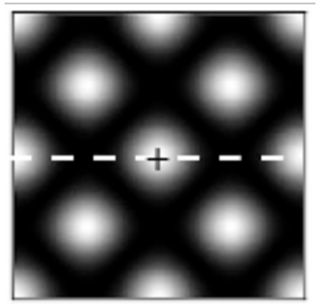
J. Sebby-Strabley, M. Anderlini, P.S. Jessen, and J.V. Porto, Phys Rev. A 73, 033605 (2006)





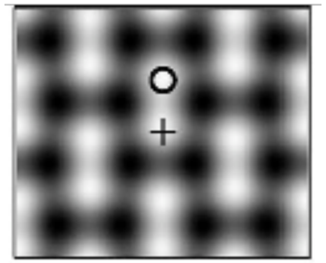
$$V(\mathbf{x}) = V_0 \left[\cos(kx)^2 + \cos(ky)^2\right]$$

- in plane polarization
- cross terms disappear

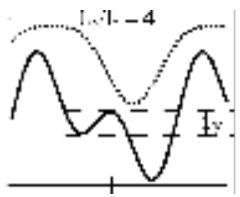


$$V(\mathbf{x}) = V_0 \left[\cos(kx) + \cos(ky)\right]^2$$

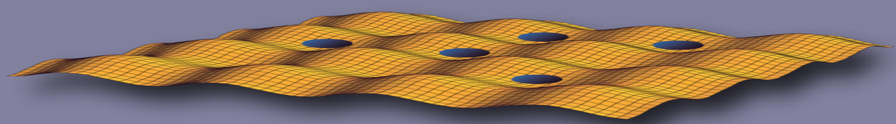
- polarization along z-axes
- lattice with cross terms



- combined lattice
- lattice of double wells



Many body Hamiltonian



Microscopic Hamiltonian

Many-body Hamiltonian

- pseudo-potential approximation

- field operator $\psi(x), \psi^\dagger(x)$

$$H = \int dx\, \psi^+(x) \left(-\frac{\hbar^2}{2m}\Delta + V(x) \right) \psi(x) + \frac{g}{2} \int dx\, \psi^+(x)\psi^+(x)\psi(x)\psi(x)$$

optical lattice

$$g = \frac{4\pi\hbar^2 a_s}{m}$$

: interaction strength of the Pseudo potential

Derivation of effective low energy theory:

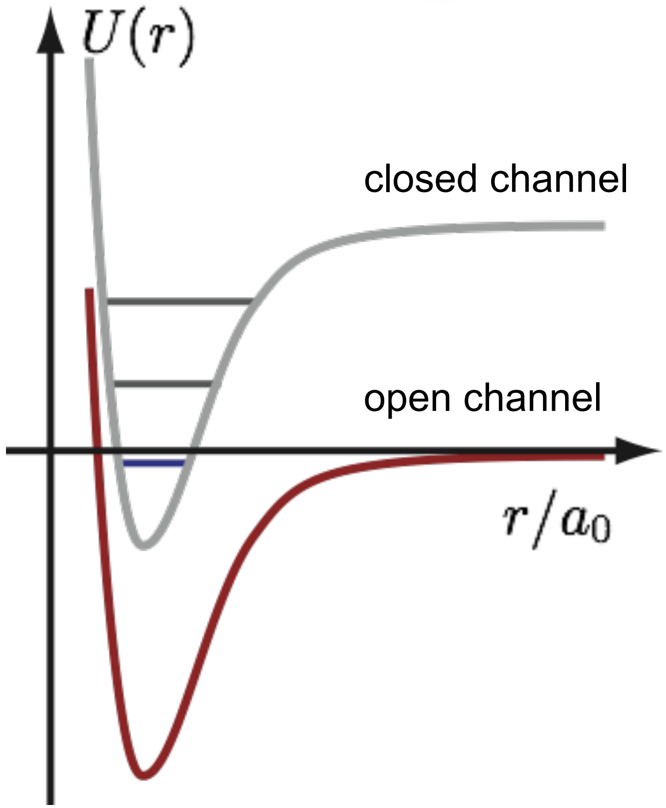
D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Rev. Lett. 81, 3108 (1998)

(i) Solve the single particle problem in an optical lattice

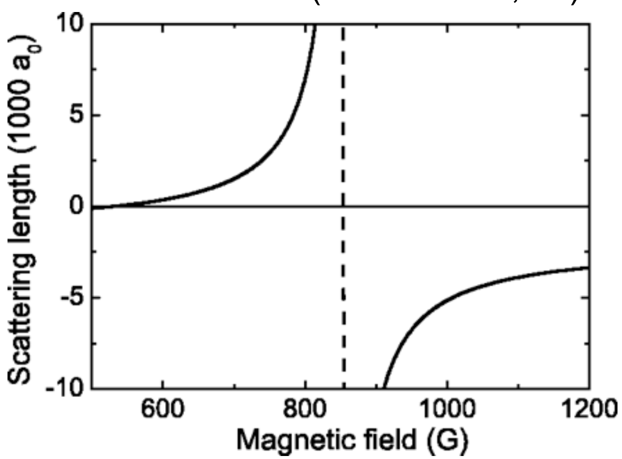
(ii) Add the interaction as perturbation

→ Hubbard model for Fermions and bosons

Magnetic Feshbach resonance



(Grimm et al., 03)



Feshbach resonances

- two interal states of the atoms:
    - open channel
    - closed channel (only virtually excited)

- bound state close to the continuum of the open channel

- tuning of the energy of the molecular state via magnetic field or Raman transition

$$a_{\rm eff} = a_s\left(1 + \frac{\Delta\nu}{E - E_{\rm res}}\right)$$

$\Delta\nu$: width of the resonance

$\nu = E - E_{\rm res}$: detuning

Single particle problem

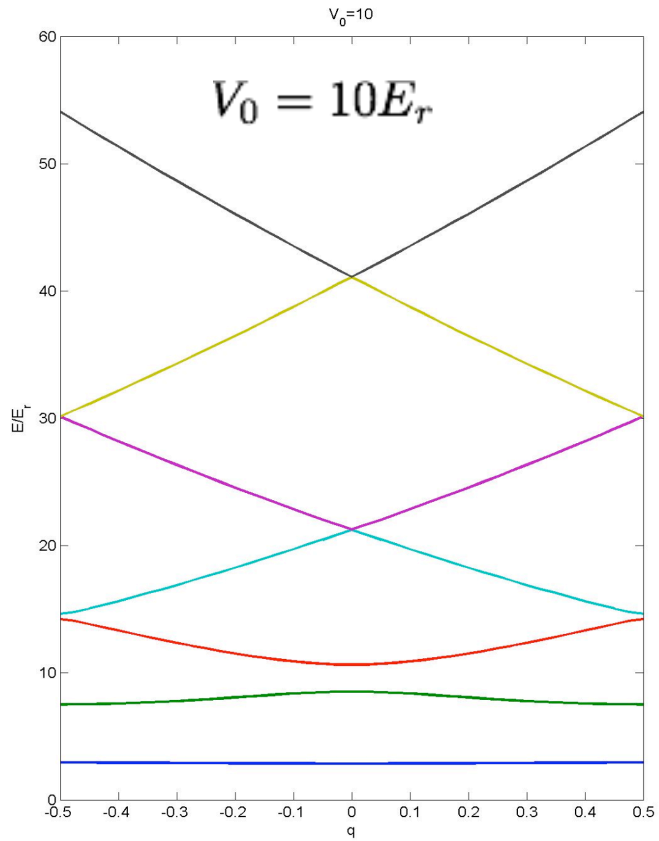
Hamiltonian

- particle in a periodic potential

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

lattice vector

$$V(\mathbf{r} + \mathbf{R}_i) = V(\mathbf{r})$$



- Bloch wave functions

$$\phi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{kr}} u_{n,\mathbf{k}}(\mathbf{r})$$

$E_n(\mathbf{k})$: energy dispersion

$\mathbf{k}$: quasi momentum within first Brillouine zone

- Bloch theorem

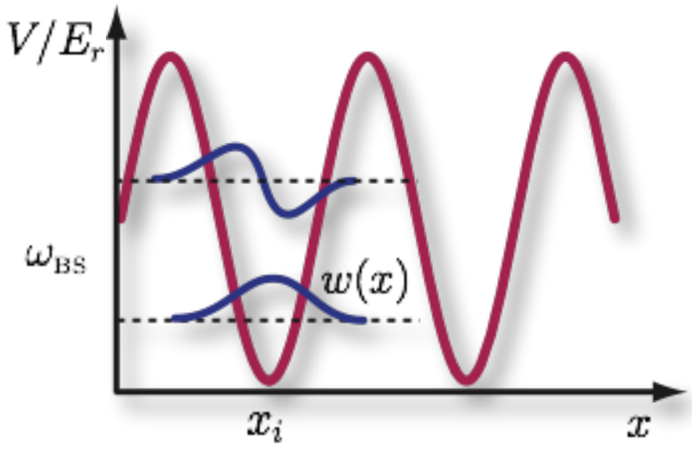
$$\phi(\mathbf{r} + \mathbf{R}_i) = e^{i\mathbf{kr}} \phi(\mathbf{r})$$

Wannier functions

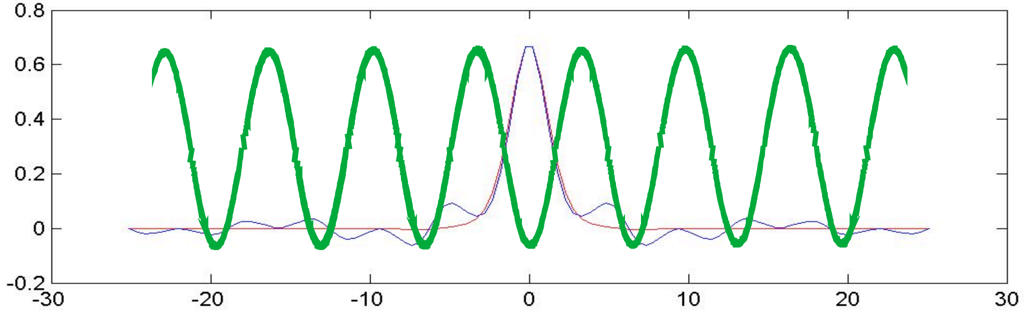
Wannier functions

- localized wave function at each lattice site

$$w_n(\mathbf{r}-\mathbf{R}_i) = \int \frac{d\mathbf{k}}{v_0} e^{-i\mathbf{k}\mathbf{R}_i} \phi_{n,\mathbf{k}}(\mathbf{r})$$



- not uniquely defined

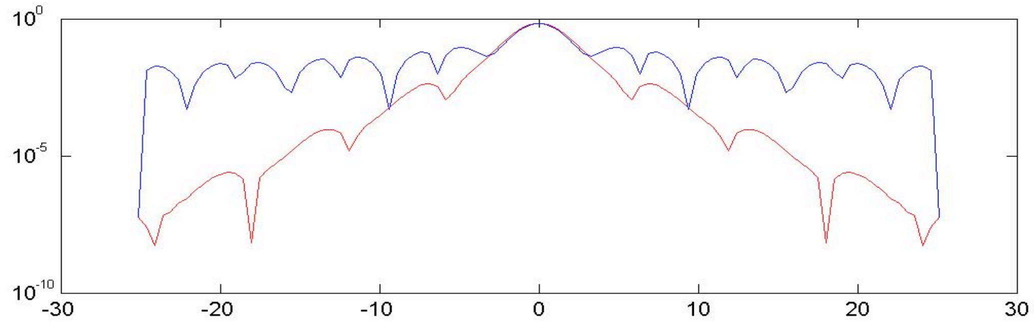


Strong optical lattice

- approximation as harmonic wave functions $|x| \ll a$

$$w(x) \approx \sqrt{\frac{1}{\pi^{1/2} a_0}} \exp\left(-\frac{x^2}{2a_0^2}\right)$$

$$a_0 = \sqrt{\frac{\hbar}{m\omega_0}} = \left(\frac{\hbar^4}{m^2 V_0 E_r}\right)^{1/4}$$



- different behavior on long distances

Microscopic Hamiltonian

Hubbard model

- express the bosonic field operator in terms of Wannier functions

$$\psi(\mathbf{r}) = \sum_{i,n} w_n(\mathbf{r} - \mathbf{R}_i) b_{n,i}$$

creation/annihilation operator for particles at site *i* in band *n*

$$H = \int dx\, \psi^+(x) \left(-\frac{\hbar^2}{2m} \Delta + V(x) \right) \psi(x) + \frac{g}{2} \int dx\, \psi^+(x) \psi^+(x) \psi(x) \psi(x)$$

- restriction to lowest bloch bands
- only largest terms

$$H_{\mathrm{BH}} = -J \sum_{\langle ij \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i$$

t

hopping energy

interaction energy

Bose-Hubbard Model

Bose-Hubbard model (Fisher et al PRB '81)

$$H_{\rm BH} = -J\sum_{\langle ij\rangle} b_i^\dagger b_j + \frac{U}{2}\sum_i b_i^\dagger b_i^\dagger b_i b_i - \sum_i \mu_i b_i^\dagger b_i$$

hopping energy

interaction energy

chemical potential and trapping

Coupling parameters in 1D

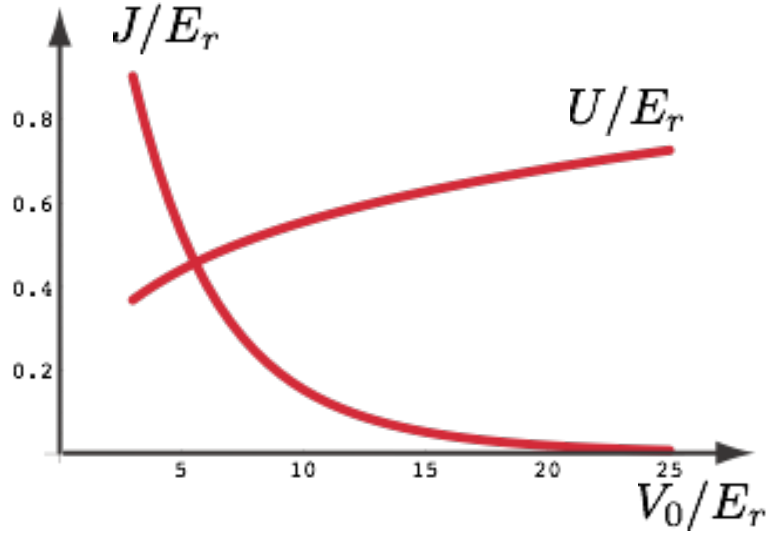
- onsite-interaction

$$U = \sqrt{2/\pi}\,\hbar\omega_\perp \frac{a_s}{a_0}$$

- hopping energy

$$J = \frac{4}{\sqrt{\pi}}\sqrt{VE_r}\left(\frac{V}{E_r}\right)^{1/4} e^{-2\sqrt{V/E_r}}$$

- validity $U \ll \hbar\omega_0$

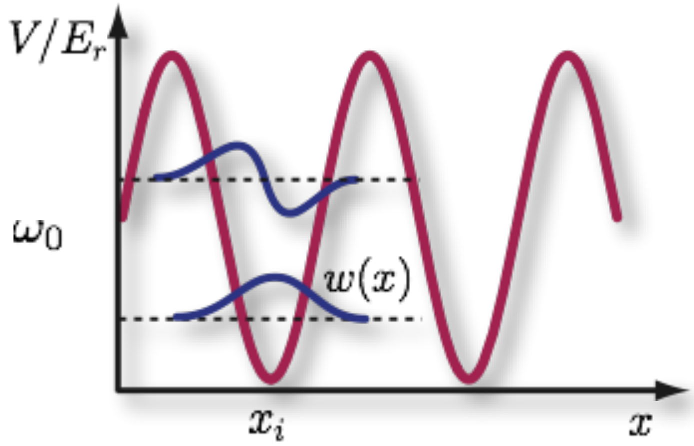


Bose-Hubbard Model

Weak interactions

- the mixing of different Bloch-bands is suppressed for weak interactions

$$U \ll \hbar\omega_0 \quad \Longleftrightarrow \quad a_s < a_0$$



Next-nearest-neighbor hopping

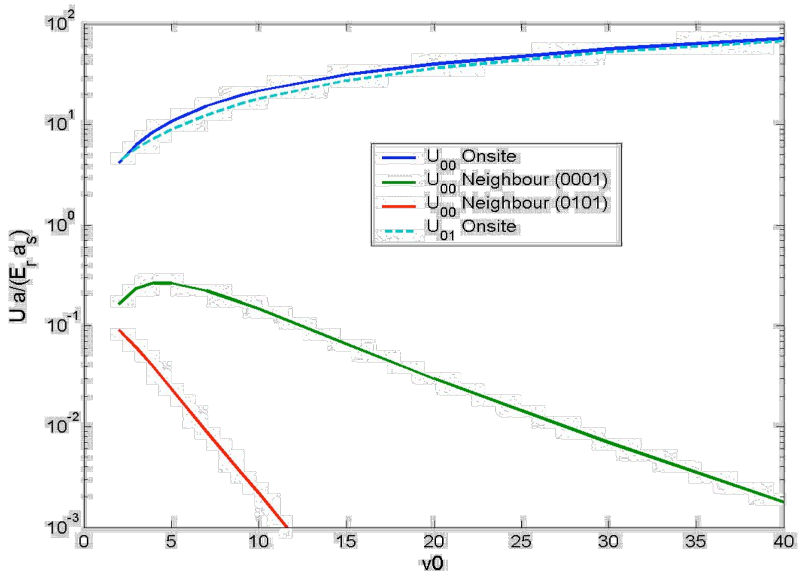
- small in the tight binding limit, but have to be included fore weak opitcal lattices

$$V_0 \lesssim E_r$$

Nearest-neighbor interaction

- nearest-neighbor interactions are present, but are suppressed due to the decay of the wannier functions

$$U_1 \sim \int d\mathbf{r} |w(\mathbf{r})|^2 |w(\mathbf{r}-\mathbf{R}_i)|^2$$



Energy scales

Characteristic parameters

- wave length

$$\lambda \sim 1000\mathbf{nm}$$

- lattice spacing

$$a \sim 500\mathbf{nm}$$

- temperatures

$$T_{\mathrm{BEC}} \approx 1\mu\mathbf{K}$$

$$T_{\min} \sim 1\mathbf{nK}$$

$$(1\mathbf{Hz} \equiv 50\mathbf{pK})$$

$$E_r = \frac{2\pi^2\hbar^2}{m\lambda^2} \approx 9\mathbf{kHz}$$

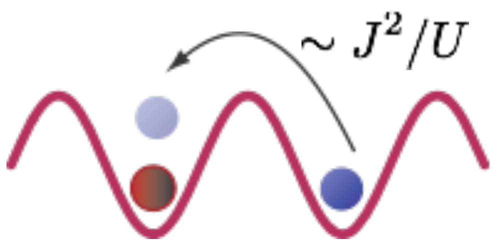
$$\approx 430\mathbf{nK}$$

$$U \approx 0.5 E_r$$

$$J \approx 0 - 0.5 E_r$$

Exchange coupling

- anti-ferromagnetic coupling in a fermionic Hubbard model



- effective interaction can become extremely small

- extremely challenging for temperature and stability

Phase diagram

Superfluid $J \gg U$

- weakly interacting Bose-Einstein condensate

$$\phi_{\mathrm{BEC}} \sim \left(\sum_i b_i^\dagger\right)^N |0\rangle$$

- linear excitation spectrum
- off-diagonal long-range

$$\langle \psi(\mathbf{r})\psi^\dagger(0)\rangle \rightarrow n_0$$

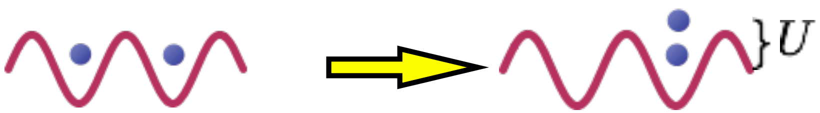
- delocalized atoms
- poisson statistic for number of atoms per lattice site

Mott insulator $U \gg J$

- commensurate filling
- zero temperature phase
- fixed particle number per lattice site

$$\phi_{\mathrm{Mott}} = \prod_i b_i^\dagger |0\rangle$$

- excitation gap $\Delta \sim U$



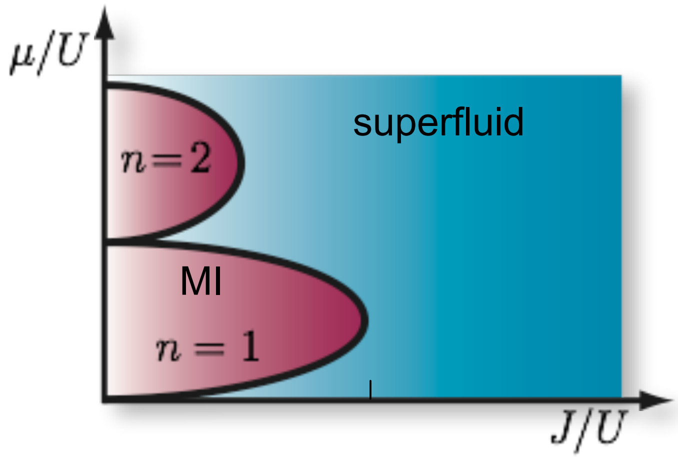
- localized particles
- integer particles per lattice site

Phase diagram

Quantum phase transition

Mott insulator

- commensurate filling
- gapped phase
- incompressible



Superfluid

- long-range order
- finite superfluid stiffness
- linear excitation spectrum

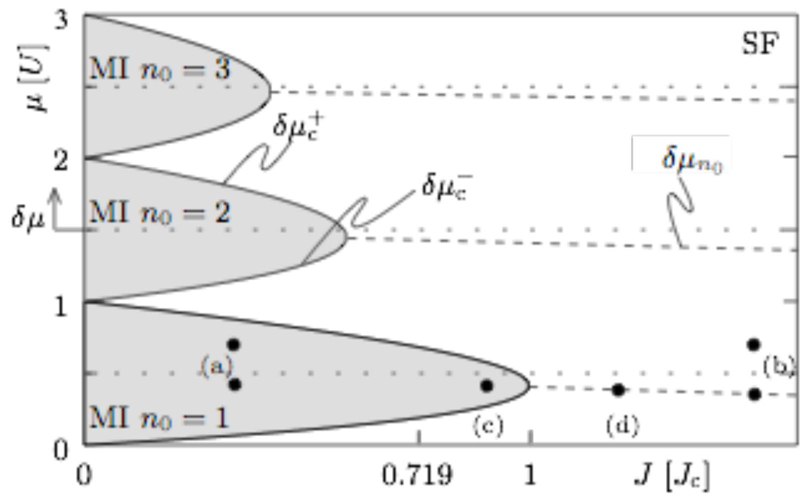
Mean-field theory $(d = \infty)$

- critical value

$$U/J\Big|_{\mathrm{S-MI}} = z\left(n + \sqrt{n+1}\right)^2$$

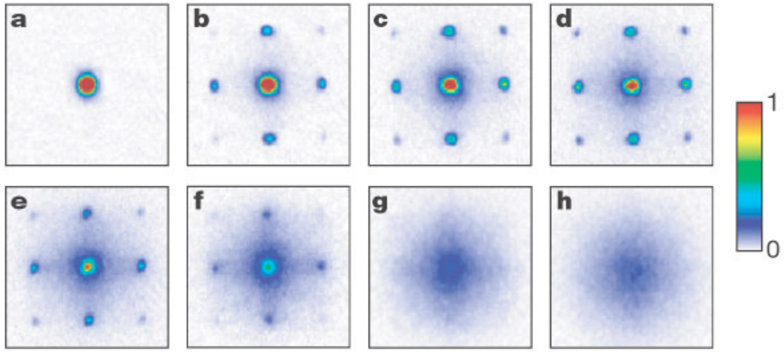
- qualitative correct even at low dimensions

- no particle-hole fluctuations in the Mott insulator



Experiments

Long-range order:

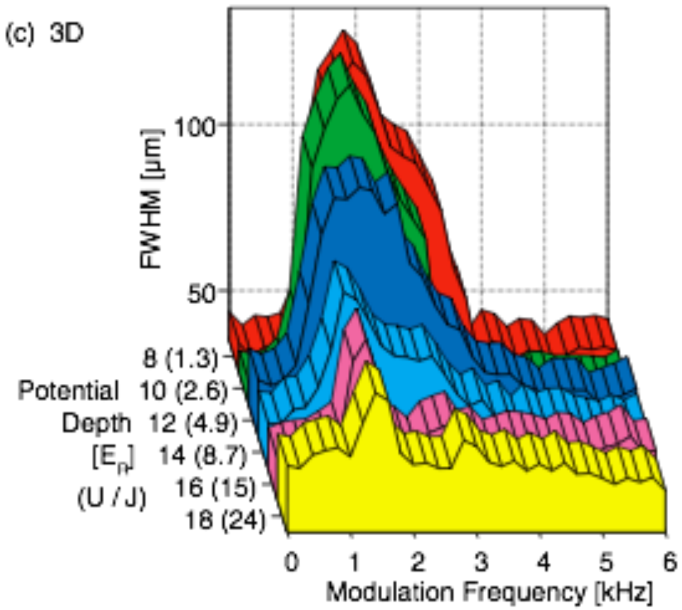


(Greiner et al., 02)

Disappearance of coherence for strong optical lattices (Greiner et al. '02)

$$\frac{V}{E_r} > 13$$

Structure factor



(Esslinger et al., 04)

Appearance of well defined two particle excitations

Mean-field theory

Bose-Hubbard Model

Bose-Hubbard model (Fisher et al PRB '81)

$$n_i = b_i^\dagger b_i$$

$$H_{\text{BH}} = -J \sum_{\langle ij \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i$$

hopping energy | interaction energy | chemical potential

Mean-field ansatz:

- mean-field value: $\psi = \langle b_i \rangle$ — number of particles is not fixed

- small fluctuations around mean field: $b_i = \psi + (b_i - \psi)$

- $b_i^\dagger b_j = \psi^* b_j + b_i^\dagger \psi - |\psi|^2$ ⇒

local Hamiltonian

$$H_{\text{BH}} = \sum_i H_i$$

Mean-field theory

Local Hamiltonian

number of nearest-neighbors

$$H_i = -2zJ\left(\psi^* b_i + \psi b_i^\dagger - |\psi|^2\right) + \frac{U}{2} n_i (n_i - 1) - \mu n_i$$

- ground state is a product state

$$\rho = \prod_i \rho_i$$

local density matrix

- self-consistency

$$\psi = \langle b_i \rangle$$

- zero-temperature

$$\rho_i = |f\rangle\langle f|$$

$$|f\rangle = \sum_n f(n)|n\rangle$$

Long-Range order

- order parameter

condensate density

$$\langle b_i b_j^\dagger \rangle \longrightarrow n_0$$

- mean-field result

$$\langle b_i b_j^\dagger \rangle = |\psi|^2 \equiv n_0$$

- the mean-field ψ breaks the U(1) symmetry
- order parameter of the quantum phase transition

Mean-field theory

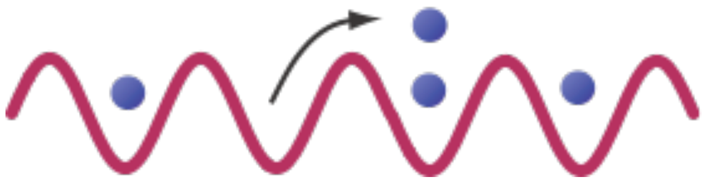
Mott Insulator

- localized particles
- integer particles
  per lattice site

$$|f\rangle = |n_0\rangle$$

- realistic Mott-Insulator
  has particle-hole fluctuations

"mean-field" Mott insulator



particle-hole fluctuations

Superfluid phase

- off-diagonal long range order

$$\psi \neq 0$$

- weak interactions:
  - locally a coherent state
  - condensate involves
    all particles

$$|f\rangle = \exp\left(-|\psi|^2/2\right) \exp\left(\psi b^\dagger\right)$$

Mean-field theory

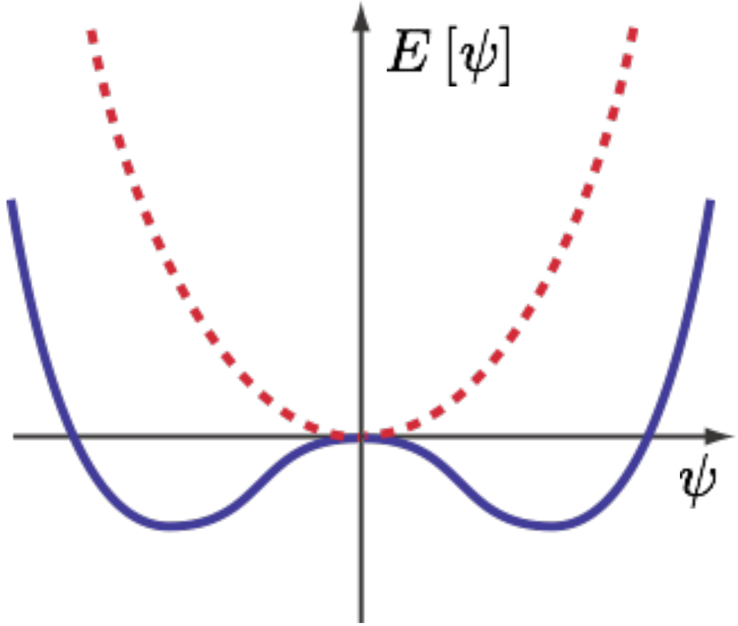
Phase transition

- appearance of long-range order $\psi \neq 0$

- Ansatz across the transition

$$|f\rangle = \sqrt{1-2\epsilon^2}|n_0\rangle + \epsilon\left(e^{i\phi}|n_0-1\rangle + e^{-i\phi}|n_0+1\rangle\right)$$

$$\psi = 2e^{i\phi}\epsilon\sqrt{1-2\epsilon^2}$$



- energy

$$\frac{E[\psi]}{U} = \left[1 - \frac{zJ}{U}\left(\sqrt{n_0} + \sqrt{n_0+1}\right)^2\right]|\psi|^2 + \beta|\psi|^4$$

phase transition at the change of the sign

Phase diagram

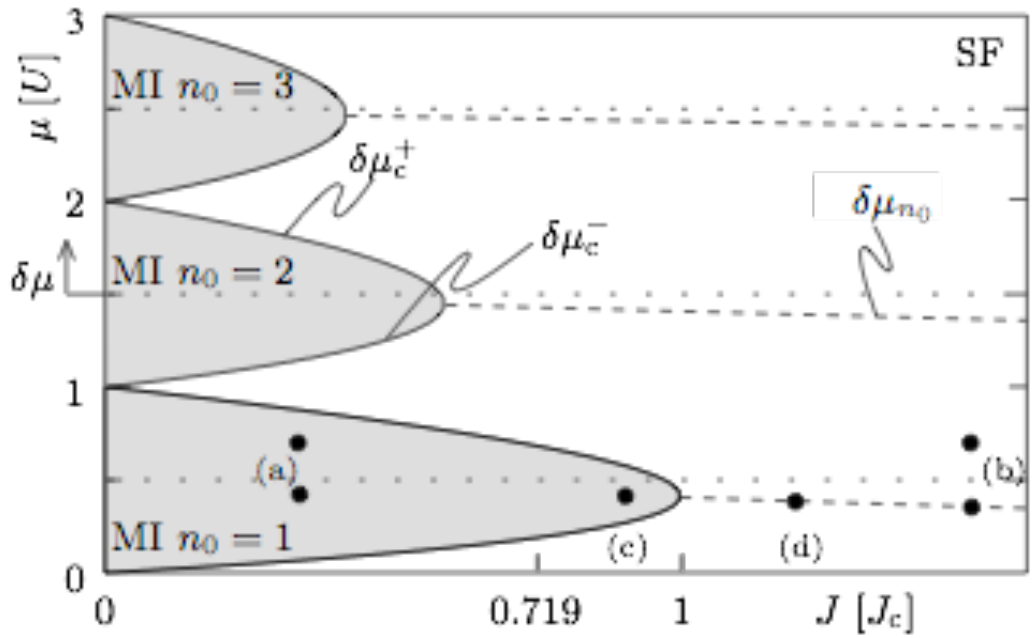
Quantum phase transition

- critical value

$$U/J\Big|_{\mathrm{S-MI}} = z\left(n+\sqrt{n+1}\right)^2$$

- qualitative correct even at low dimensions

- no particle-hole fluctuations in the Mott insulator



Effective field theory

Effective field theory

Partition function

- $$Z_B = \mathrm{Tr}\left[\exp\left(-\frac{H_{BH}}{T}\right)\right]$$

propagator in imaginary time

$$U = \exp\left(-\frac{itH_{BH}}{\hbar}\right)$$
$$t = -i\tau$$

- path integral formulation

$$Z_B = \int \mathcal{D}\left[b_i(\tau)\right]\mathcal{D}\left[b_i^\dagger(\tau)\right]\exp\left(-\int_0^{1/T} d\tau \mathcal{L}_B\right)$$

- Lagrangian of the Bose-Hubbard model

$$\mathcal{L}_B = \sum_i \left[b_i^\dagger \partial_\tau b_i - \mu b_i^\dagger b_i + \frac{U}{2} b_i^\dagger b_i^\dagger b_i b_i\right] - J\sum_{\langle i,j\rangle}\left(b_i^\dagger b_j + b_i b_j^\dagger\right)$$

$$= \sum_i b_i^\dagger \partial_\tau b_i + H_{BH}$$

opposite sign due to imaginary time

Effective field theory

Hubbard Stratonovich Transformation

- Decoupling of the non-local term

$$\exp\left(\int_0^{1/T} d\tau \sum_{i,j} b_i^\dagger A_{i,j} b_j\right)$$

non-local hopping term

$$A_{i,j} = \begin{cases} J & \langle i,j \rangle \\ 0 & \text{else} \end{cases}$$

$$= \int \mathcal{D}\left[\psi_i(\tau)\right] \mathcal{D}\left[\psi_i^\dagger(\tau)\right] \exp\left(-\int_0^{1/T} d\tau \left[\sum_{i,j} \psi_i^\dagger A_{i,j}^{-1} \psi_j - \sum_i \left(\psi_i b_i^\dagger + \psi_i^\dagger b_i\right)\right]\right)$$

effective field: order parameter

quadratic term

local term

Effective field theory

Integrating out the bosons

$$\mathcal{L}_{\mathrm{B}} = \mathcal{L}_0 - \sum_i \left[\psi_i b_i^\dagger + \psi_i^\dagger b_i\right] + \sum_{i,j} \psi_i^\dagger A_{i,j}^{-1} \psi_J$$

- local theory

$$\mathcal{L}_0 = \sum_i \left[b_i^\dagger \, \partial_\tau b_i - \mu b_i^\dagger b_i + \frac{U}{2} b_i^\dagger b_i^\dagger b_i b_i\right]$$

- Greens function

$$G(i\omega_s) = -\langle T_\tau b_i(\tau) b_i^\dagger(0)\rangle = \frac{n_0 + 1}{i\omega_s + \mu - U n_0} - \frac{n_0}{i\omega_s + \mu - U(n_0 - 1)}$$

Effective field theory

Effective theory

- continuum limes

$$Z_{\mathrm{B}} = \int \mathcal{D}\left[\psi(\tau,\mathbf{x})\right]\left[\psi^{\dagger}(\tau,\mathbf{x})\right]\exp\left(-\int_0^{1/T} d\tau\, \mathcal{L}_{\mathrm{B}}\right)$$

- lagrangian

$$\mathcal{L}_{\mathrm{B}} = \sum_i \psi_i^{\dagger} G(\tau)\psi_i + \sum_{i,j} \psi_j^{\dagger} A_{i,j}^{-1}\psi_i + O(|\psi|^4)$$

$$\approx \int d\mathbf{x}\left[K_1\psi^{\dagger}\partial_{\tau}\psi + K_2|\partial_{\tau}\psi|^2 + K_3|\nabla\psi|^2 + r|\psi|^2 + u|\psi|^4 + \ldots\right]$$

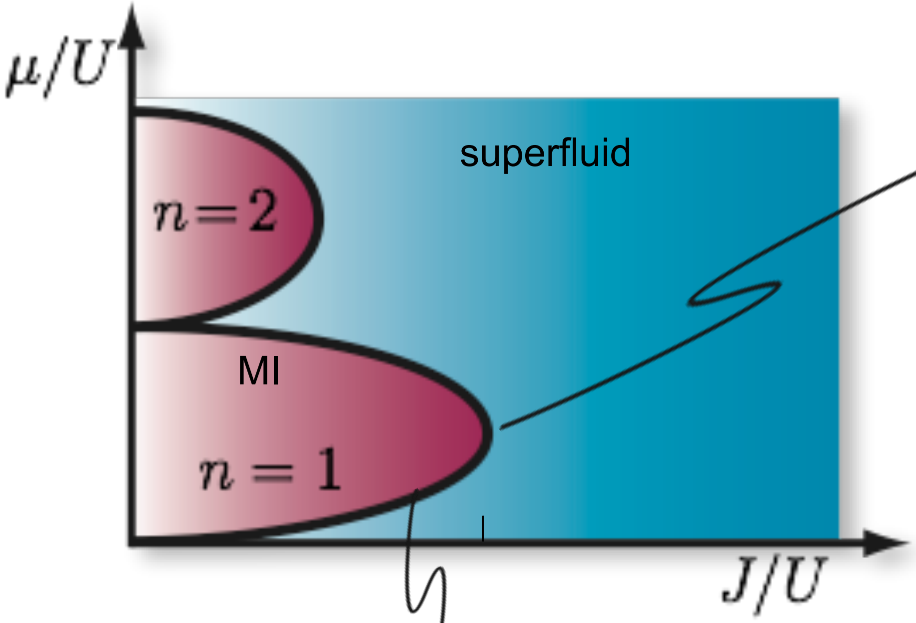
- symmetry:

$$K_1 = -\partial_{\mu} r$$

$$r = \frac{1}{zJ} + G(0)$$

change of sign determines phase transition

Effective field theory



Tip of the Lobe:

- density is continous
- leading term vaishes $K_1 = 0$
- relevant dynamics

$$K_2 |\partial_\tau \psi|^2$$

- universality class of the O(2) quantum rotor model

Lobe boundary:

- change in particle density
- relevant dynamics

$$K_1 \psi^\dagger \partial_\tau \psi$$

- universality class of the dilute Bose-gas

Bose-Einstein condensation of excitations (particles/holes) above the Mott insulator

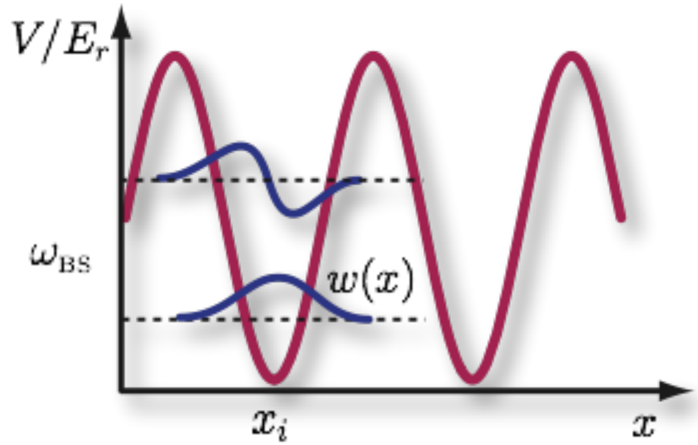
particles

holes

Conclusion

Quantum phase transition
in cold gases

- optical lattices
- realization of the Bose-Hubbard
  model

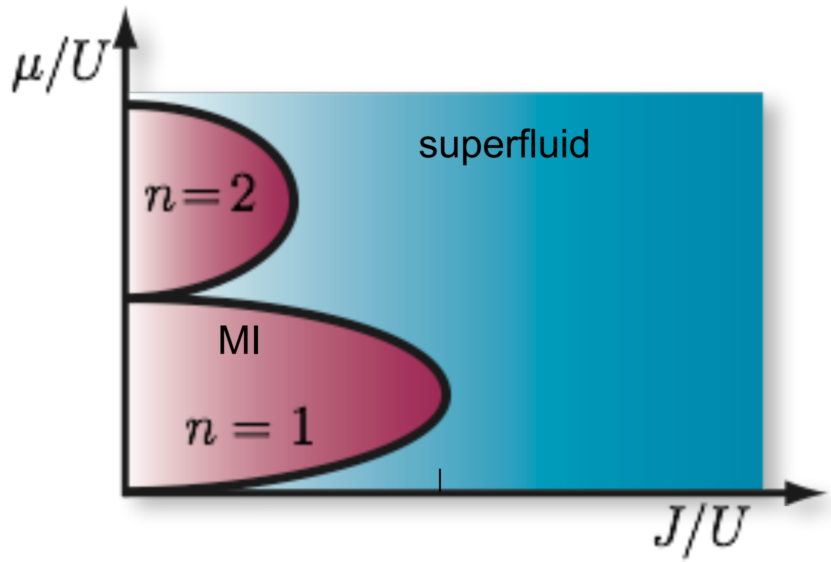


Description of the
Bose-Hubbard model

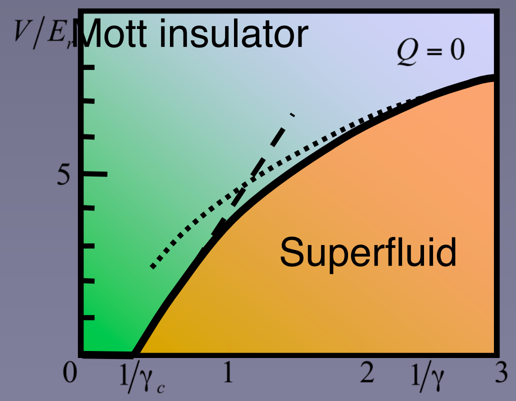
- mean-field theory

- effective theory

- 1D system



Bose-Hubbard model in 1D



Why one dimension?

Interaction potential

- pseudo potential in 1D

$$V(x) = g\delta(x) = 2\hbar\omega_\perp a_s \delta(x)$$

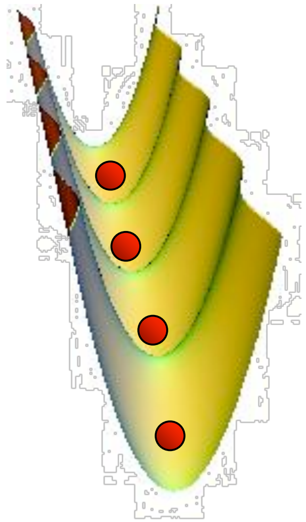
- interaction strength

$$\gamma = \frac{E_{\text{int}}}{E_{\text{kin}}} = \frac{mg}{\hbar^2 n}$$

- homogoneous system is exactly solvable (Lieb and Liniger)

  $\gamma \ll 1$: weakly interacting bosons
  - mean field theory
  - Bogoliubov theory

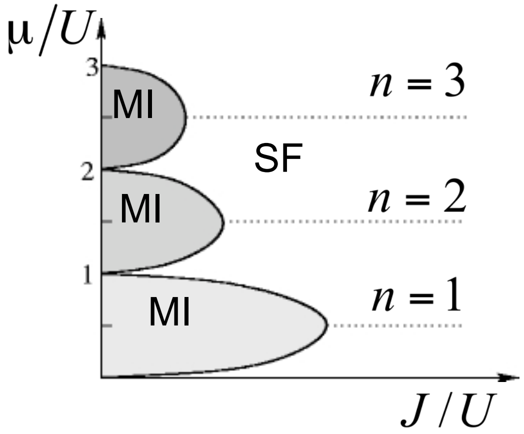
  $\gamma \gg 1$: Tonk gas limit



- increased fluctutations in 1D
- strongly interacting regime

Bose-Hubbard model

$V_0 > E_r$



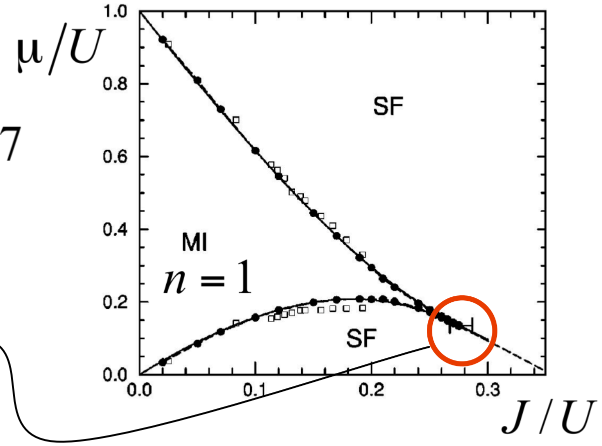
- Mean field (Krauth et al., '92)

$$U/J\big|_{\text{S-MI}} = z\left(n+\sqrt{n+1}\right)^2_{n=1} \approx 11.7$$

- With fluctuations (Kühner et al., '98)

$$U/J\big|_{\text{S-MI}} = 2C \approx 3.8$$

Kühner and Monien, '98

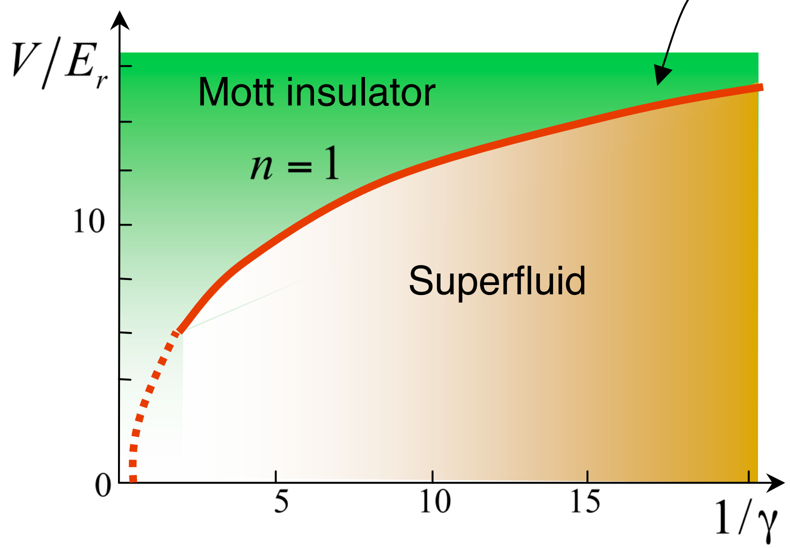


- Transition line in the γ -V phase diagram

$$U \propto \gamma$$

$$J \propto \exp\left(-2\sqrt{V/E_r}\right)$$

$$\Rightarrow \quad 4V/E_r = \ln^2\left[\frac{4\sqrt{2\pi}\,C\sqrt{V/E_r}}{\gamma}\right]$$



$\gamma \rightarrow \infty$ ⟹ Mapping to the Bose-Hubbard model not valid.

What happens in the strongly interacting Bose gas?

Hamiltonian

$$H = \int dx \left[\psi^+(x) \left(-\frac{\hbar^2}{2m} \Delta + V(x) \right) \psi(x) + \frac{g}{2} \psi^+(x) \psi^+(x) \psi(x) \psi(x) \right]$$

- interaction strength

$$g = 2\hbar\omega_T a_s \qquad \gamma = \frac{mg}{n\hbar^2}$$

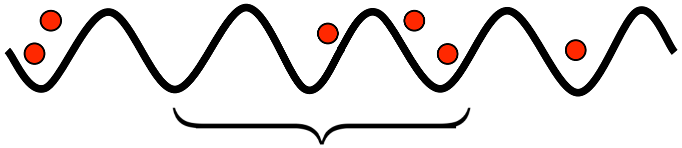
scattering length

- external potential

$$V(x) = V \sin^2(kx) + V_{\text{trap}}(x)$$

$$E_r = \frac{\hbar^2 k^2}{2m}$$

recoil energy



$$\lambda = k/2\pi$$

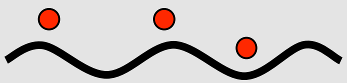
Sine Gordon model

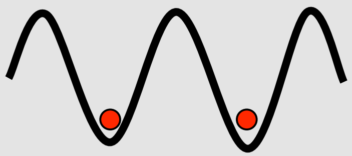
$V_0 < E_r$

Superfluid to Mott insulator quantum phase transition

Bose-Hubbard model

$V_0 > E_r$





Hydrodynamic description

$V_0 \lesssim E_r$

- Bosonic field operator in terms of a long-wavelength density- and phase-field operator θ and φ (Haldane, '81)

commutation relation

$$\psi(x) \approx \sqrt{n + \partial_x \theta/\pi}\, \exp(i\phi)$$

$$\left[\phi(x), \partial_y \theta\right] = i\pi\delta(x-y)$$

- Hamiltonian (without optical lattice)

$$H_0 = \frac{\hbar}{2\pi}\int dx \left[v_J (\partial_x \phi)^2 + v_N (\partial_x \theta)^2 \right]$$

superfluid stiffness $v_J = \pi\hbar n/m$

inverse compressability $v_N = \partial_n \mu/\pi\hbar$

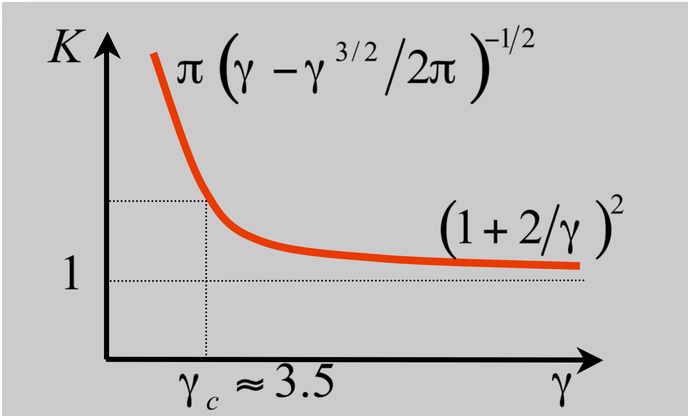
- quasi long-range order

$$\langle \psi(x)\psi^\dagger(0)\rangle \sim |x|^{-1/2K} \qquad K = \sqrt{v_J/v_N}$$

- sound velocity

$$v_s = \sqrt{v_J v_N}$$

(Lieb and Liniger,'63)

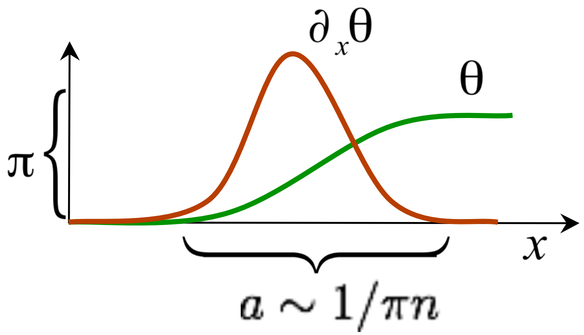


Optical lattice

$V_0 \lesssim E_r$

- Taking into account the discrete nature of the particles (Haldane, '81)

$$n(x) = \left[n + \partial_x \theta / \pi\right] \left\{1 + 2\sum_{s=1}^{\infty} \cos\left(2s\theta + 2s\pi n x\right)\right\}$$



- Interaction Hamiltonian

$$H_V = \frac{V}{2} \int dx\, n(x) \cos\frac{4\pi x}{\lambda}$$

Reduction to the relevant term

Commensuration

$$Q = 2\pi\left(n - 2/\lambda\right)$$

- Perturbation from a weak optical lattice

$$H_V = \frac{Vn}{2} \int dx \cos\left[2\theta(x) + Q x\right]$$

Sine-Gordon model

$V_0 \lesssim E_r$

$$H = \frac{\hbar v_s}{\pi}\int dx\left\{\frac{1}{2}\left[K(\partial_x\phi)^2 + \frac{1}{K}(\partial_x\theta)^2\right] - \frac{Q}{2K}\partial_x\theta + \frac{u}{a^2}\cos(2\theta)\right\}$$

- Three parameters

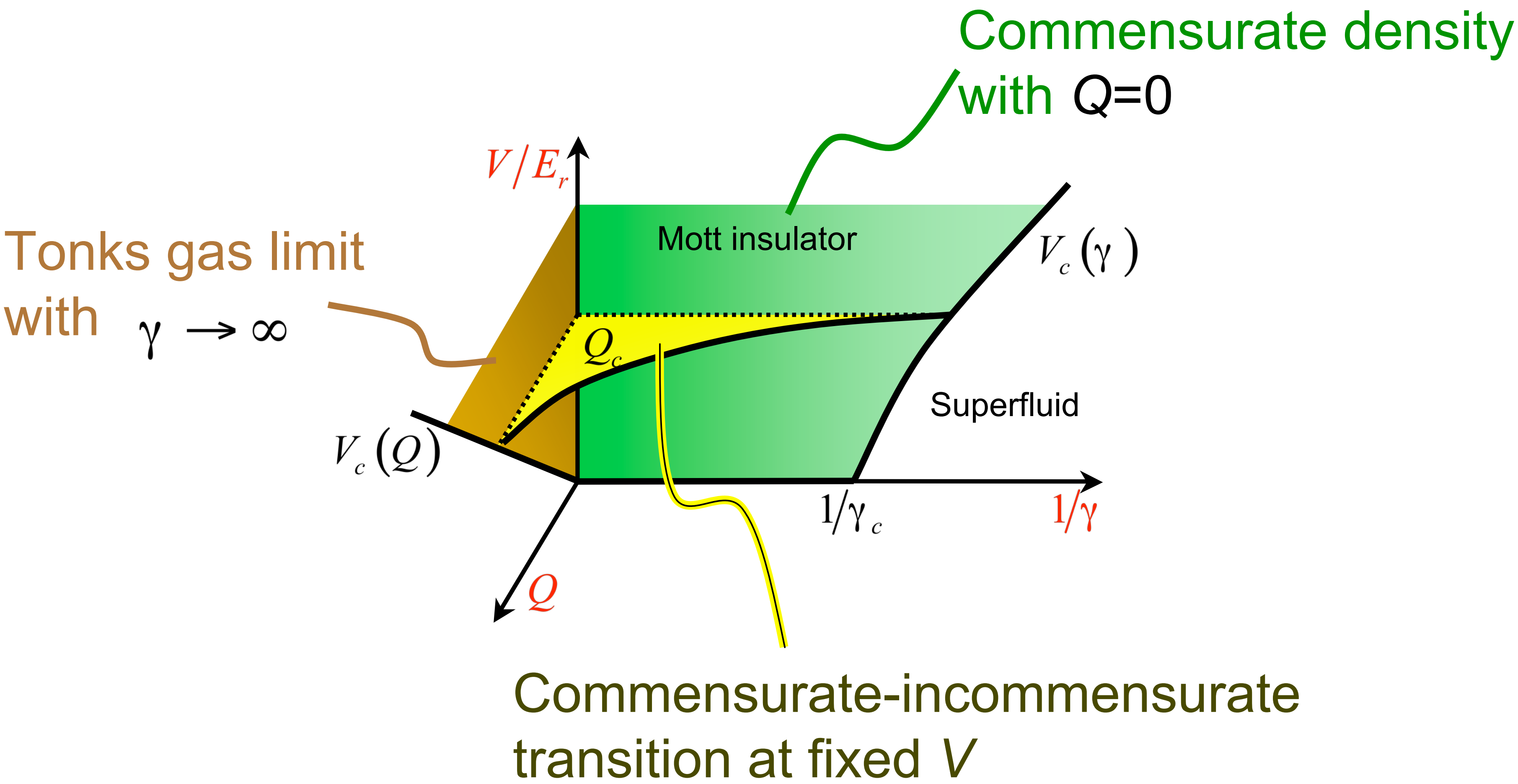
$K(\gamma)$ →	Measures strength of fluctuations in the density field θ
$Q = 2\pi\,(n - 2/\lambda)$ →	Induces a shift in the chemical potential and drives the system away from commensurability
$u = \pi a^2 n V / 2\hbar v_s$ →	Strength of the optical lattice

- Exactly solvable field theory
  - massive spinless fermions
  - U(1) symmetric Thirring model in a magnetic field

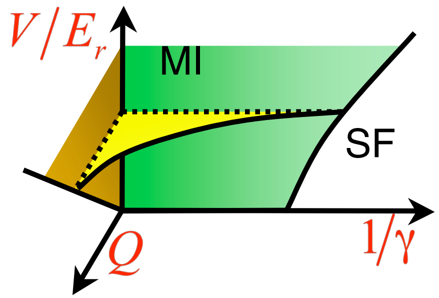
(Coleman, '75; Wiegmann, '78; Japardize et al., '84;
Pokrovsky et al., '79; Kehrein, '99)

Phase diagram

$V = E_r$

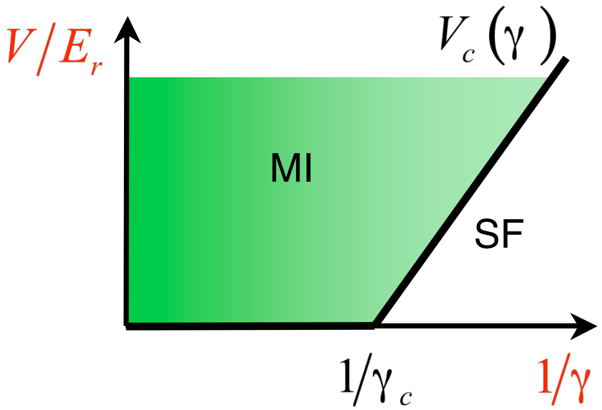


Phase diagram



Commensurate density $Q=0$

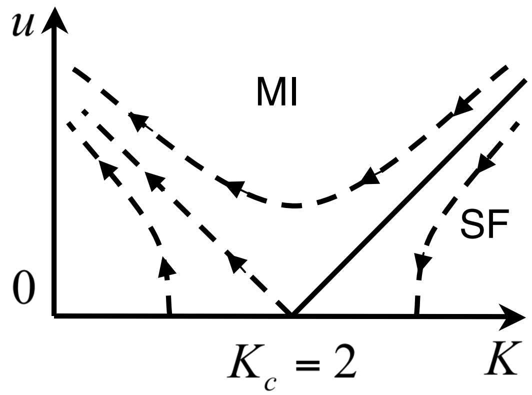
• Instability in the sine-Gordon model (Coleman, '75)



$$K_c(\gamma)=2$$

$$\Downarrow$$

$$\gamma_c \approx 3.5$$



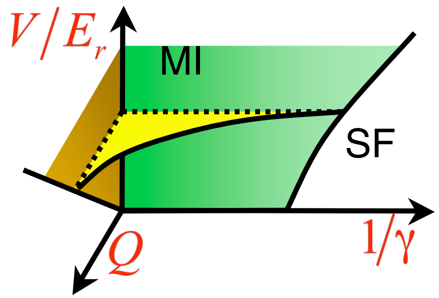
• Kosterlitz-Thouless universality class

$$K_c(u)=2(1+u)$$

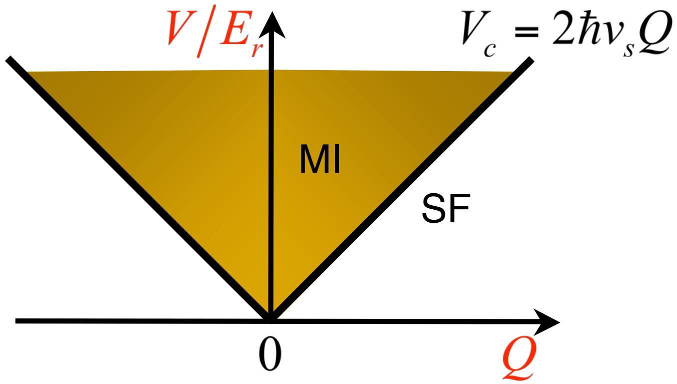
$$\Downarrow$$

$$V_c(\gamma)\approx E_r\left(\gamma^{-1}-\gamma_c^{-1}\right)/5.5$$

Phase diagram

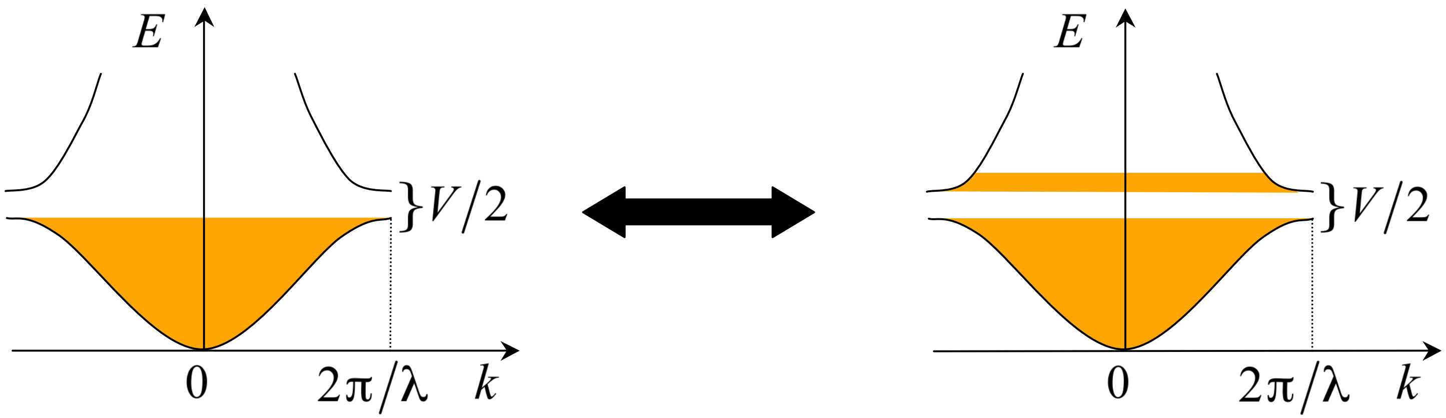


Tonks gas $\gamma \to \infty$



- The bosons wave function maps to the wave function of free fermions in periodic potential (Girardeau,'60)

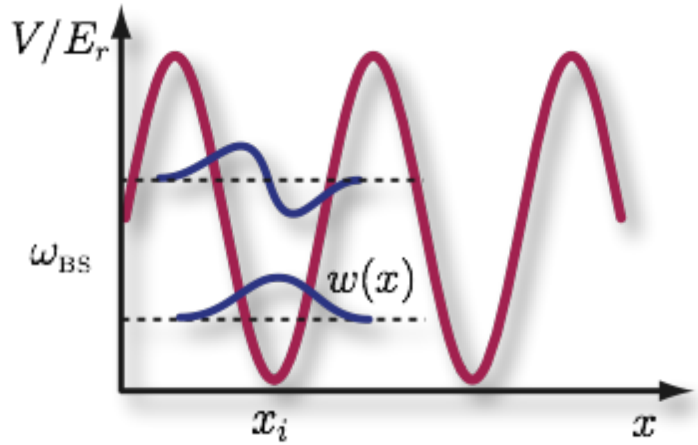
- At commensurate filling the fermions are a standard band insulator with a single particle gap $2\Delta = V/2$



Conclusion

Quantum phase transition
in cold gases

- optical lattices
- realization of the Bose-Hubbard model



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- effective theory

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