Introduction to Quantum Information

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1 Quantum Mechanics

1.1 Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle \tag{1}$$

$|\psi(t)\rangle$ State vector; element of Hilbert space $\mathcal{H}$ (= vector space over complex numbers, with scalar product)

$\hat{H}$ Hamilton operator (self-adjoint (hermitean) operator on a Hilbert space $\mathcal{H}$)

Formal solution for $\hat{H}$ time-independent

$$\begin{aligned}&\left|\psi(t)\right\rangle = U(t)\left|\psi(0)\right\rangle \quad \text{with}\\ &U(t) = \mathrm{e}^{-\frac{i}{\hbar}\hat{H}t}\end{aligned} \tag{2}$$

$U(t)$ Time-evolution operator. $U(t)$ is unitary:

$$U(t)U(t)^\dagger = \mathrm{e}^{-\frac{i}{\hbar}\hat{H}t}\mathrm{e}^{\frac{i}{\hbar}\hat{H}t} \overset{\text{hermiticity}}{=} \mathbb{1} = U(t)^\dagger U(t) \tag{3}$$

1.2 Measurements: Positive Operator Valued Measure (POVM)

Def. (POVM): A set of positive operators $\{\hat{E}_\mu\}$, i.e. $\hat{E}_\mu \geq 0$ with $\sum_\mu \hat{E}_\mu = \mathbb{1}$ is a POVM. $\hat{E}_\mu$ is called "POVM element".

The state of ϱ after the outcome i occurs becomes

$$\varrho_i = \frac{U_i\sqrt{\hat{E}_i}\varrho\sqrt{\hat{E}_i}U_i^\dagger}{\mathrm{tr}\left(\hat{E}_i\varrho\right)}. \tag{4}$$

- Probability to find outcome μ is $p_\mu = \mathrm{tr}\left(\hat{E}_\mu\varrho\right)$.
- POVM can be seen as a v. Neumann measurement on a higher-dim. Hilbert space

Example: Unambiguous state discrimination

Consider two non-orthogonal pure states

$$\begin{aligned}&\left|u\right\rangle = \cos\alpha\left|0\right\rangle + \sin\alpha\left|1\right\rangle\\ &\left|v\right\rangle = \sin\alpha\left|0\right\rangle + \cos\alpha\left|1\right\rangle, \quad \text{where} \quad \left\langle u\,|\,v\right\rangle = \sin(2\alpha)\end{aligned} \tag{5}$$

Task: distinguish $|u\rangle$ and $|v\rangle$ without making mistake:

$$\begin{aligned}&\hat{E}_u = \frac{\mathbb{1} - |v\rangle\langle v|}{1+\sin(2\alpha)} \text{ detects } u \text{ with certainty}\\ &\hat{E}_v = \frac{\mathbb{1} - |u\rangle\langle u|}{1+\sin(2\alpha)} \text{ detects } v \text{ with certainty}\\ &\hat{E}_? = \mathbb{1} - \hat{E}_u - \hat{E}_v \text{ inconclusive result}\end{aligned} \tag{6}$$

2 Qubits

- classical information processing with binary alphabet: variable can take values 0 or 1, "bit"
- quantum information processing: information carriers are quantum states, basis states $|0\rangle$ and $|1\rangle$, but the sate $|\psi\rangle$ can be in any superposition.
- Realization: spin-$\frac{1}{2}$ particle, linearly polarized photon, atom in ground/exited state, etc.

General normalized pure state of a qubit:

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{i\phi}\sin\frac{\theta}{2}\,|1\rangle\,, \quad \text{with} \quad \langle i\,|\,j\rangle = \delta_{ij} \tag{7}$$

and $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Basis states:

$$\begin{aligned} |0\rangle &\equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ |1\rangle &\equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{8}$$

2.1 The density matrix

For pure states: $|\psi\rangle = \sum_i c_i\,|i\rangle$ with $\langle i\,|\,j\rangle = \delta_{ij}$, $\sum_i c_i c_i^* = 1$ (normalization)

Representation of pure states as density matrix:

$$\varrho_{\text{pure}} = \big|\psi\big\rangle\big\langle\psi\big| \tag{9}$$

Mixed quantum states, i.e. mixture of projectors onto pure states:

$$\varrho_{\text{mixed}} := \sum_j p_j \big|\psi_j\big\rangle\big\langle\psi_j\big| \;\; \text{with} \;\; p_i \geq 0, \;\; \sum_i p_i = 1 \tag{10}$$

Interpretation: A source emits state $|\psi_i\rangle$ with probability p_i, or ϱ describes a subsystem of a larger Hilbert space:

- $\varrho = \varrho^\dagger$
- $\mathrm{tr}(\varrho) = 1$ (normalization)
- $\varrho \geq 0$ (ϱ is positive semidefinite, i.e. it has nonnegative eigenvalues)
- Only for pure states: $\varrho^2 = \varrho$ (here ϱ is a projector)

Density operator for a pure qubit:

$$\varrho_{\text{pure}} = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix} \quad (11)$$

General mixed qubit state:

$$\varrho_{\text{mixed}} = \begin{pmatrix} \varrho_{00} & \varrho_{01} \\ \varrho_{01}^* & \varrho_{11} \end{pmatrix}, \quad (12)$$

with ϱ_{00}, ϱ_{11} real. 3 real parameters describe ϱ fully (remember: $\mathrm{tr}(\varrho) = \varrho_{00}+\varrho_{11} = 1$).

2.2 The Bloch sphere

Decompose ϱ into $\mathbb{1}$, σ_x, σ_y, σ_z as

$$\varrho = \frac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right), \quad (13)$$

with the (real) Bloch vector

$$\vec{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} \quad (14)$$

and the vector of Pauli matrices

$$\vec{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}. \quad (15)$$

Pauli matrices:

$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (16)$$

It follows

$$\begin{aligned} \varrho &= \begin{pmatrix} \frac{1}{2}+\frac{1}{2}s_z & \frac{1}{2}\left(s_x - is_y\right) \\ \frac{1}{2}\left(s_x + is_y\right) & \frac{1}{2}-\frac{1}{2}s_z \end{pmatrix} \\ s_x &= 2\mathrm{Re}\varrho_{01} \\ s_y &= -2\mathrm{Im}\varrho_{01} \\ s_z &= \varrho_{00} - \varrho_{11} \end{aligned} \quad (17)$$

Bloch ball for qubits

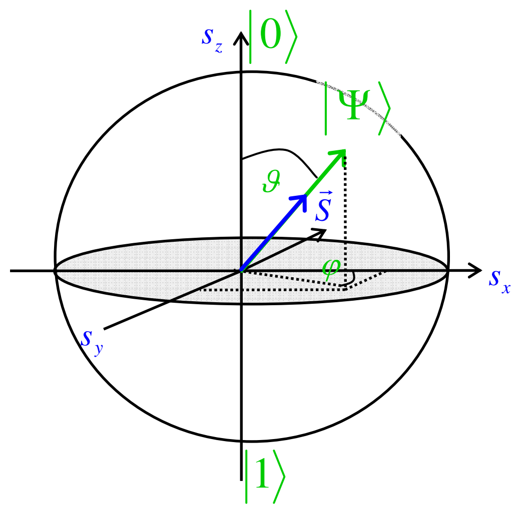


Figure 1: Bloch sphere representation a qubit in a pure (green) and a mixed (blue) state. The vector for pure states can take any point on the surface of the sphere and is represented by two parameters (ϑ and φ). For mixed states the Bloch vector $\vec{s}$ can take any point inside the sphere.

Bloch vector:

$$\vec{s} = |\vec{s}| \begin{pmatrix} \sin\vartheta\cos\phi \\ \sin\vartheta\sin\phi \\ \cos\vartheta \end{pmatrix}, \tag{18}$$

with $|\vec{s}| = 1$ for pure states, $|\vec{s}| < 1$ for mixed states.

- 3 parameters of $\vec{s}$: ϑ, ϕ, $|\vec{s}|$ determine ϱ
- pure states on surface, mixed states in interior of Bloch ball
- orthogonal states have relative angle of 180°
- every Bloch vector corresponds to physical state

3 Composite systems and entanglement

Total Hilbert space: $\mathcal{H}_A \otimes \mathcal{H}_B$

The tensor product "$\otimes$" is bilinear, i.e. $\forall\ |\psi_A\rangle$, $|\phi_A\rangle \in \mathcal{H}_A$ and $\forall\ |\psi_B\rangle \in \mathcal{H}_B$ and $c \in \mathbb{C}$:

$$\begin{aligned} & c\left(|\psi_A\rangle + |\phi_A\rangle\right) \otimes |\psi_B\rangle \\ & = |\psi_A\rangle \otimes \left(c\,|\psi_B\rangle\right) + |\phi_A\rangle \otimes \left(c\,|\psi_B\rangle\right). \end{aligned} \tag{19}$$

Example for tensor product:

$$|\psi_A\rangle \otimes |\psi_B\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_A \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_B = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}_{AB} \tag{20}$$

Notation: $|0\rangle_A \otimes |0\rangle_B \equiv |0\rangle_A\,|0\rangle_B \equiv |00\rangle_{AB} \equiv |00\rangle$
General pure state of 2 qubits:

$$|\Psi\rangle = A_{00}\,|00\rangle + A_{01}\,|01\rangle + A_{10}\,|10\rangle + A_{11}\,|11\rangle\,, \quad \sum_{ij} |A_{ij}|^2 = 1 \tag{21}$$

3.1 Entanglement

Def. (separability): A pure state is called separable iff it is a product state, i.e.

$$|\Psi\rangle_{\text{sep}} = |\psi_A\rangle \otimes |\psi_B\rangle\,. \tag{22}$$

A state which is not separable is called entangled.
Example for separable state: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right) = |0\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$
Examples for entangled states:

$$\begin{aligned} \left|\Phi^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ \left|\Psi^{\pm}\right\rangle &:= \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right), \end{aligned} \tag{23}$$

which are the Bell states (orthogonal entangled bases of 2×2 dim Hilbert space).

Given $|\psi\rangle$, is it separable or entangled?
Use "Schmidt decomposition": $\dim \mathcal{H}_A = d_A$, $\dim \mathcal{H}_B = d_B$, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, any state $|\psi\rangle \in \mathcal{H}_{AB}$ can be written as

$$|\psi\rangle = \sum_{k=1}^{r} a_k\,|e_k\rangle_A \otimes |f_k\rangle_B\,, \tag{24}$$

with Schmidt coefficients a_k real, positive, $\sum_k a_k^2 = 1$ and $\langle e_i\,|\,e_j\rangle_A = \delta_{ij} = \langle f_i\,|\,f_j\rangle_B$, with Schmidt rank r, fulfills $1 \le r \le \min(d_A, d_B)$.

$|\psi\rangle$ is separable iff $r = 1$.
Proof: Singular value decomposition of A (see Eq. 21):
$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = USV^\dagger$. Here U, V are unitary matrices, where the columns are the Schmidt vectors ($U = (|e_1\rangle, |e_2\rangle)$, $V = (|f_1\rangle, |f_2\rangle)$, and S is a diagonal matrix containing the singular values which represent the Schmidt coefficients $\left(S = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)$.

Partial trace and the Schmidt rank

Partial trace: Given a density matrix ϱ_{AB} acting on a state of $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The partial trace over subsystem B leads to

$$\varrho_A = \mathrm{tr}_B(\varrho_{AB}) = \sum_i (\mathbb{1}_A \otimes \langle i_B|)\, \varrho_{AB}\, (\mathbb{1}_A \otimes |i_B\rangle)\,, \tag{25}$$

where ϱ_A is the reduced density matrix of subsystem A.
Partial trace of $|\psi_{AB}\rangle$ in Schmidt decomposition:

$$\begin{aligned} \varrho_A = \mathrm{tr}_B(\varrho_{AB}) &= \sum_k (\mathbb{1}_A \otimes \langle f_k|) \left(\sum_{i=1}^r a_i\, |e_i\rangle\, |f_i\rangle\right) \left(\sum_{i=1}^r a_j^*\, \langle e_j|\, \langle f_j|\right) (\mathbb{1}_A \otimes |f_k\rangle) \\ &= \sum_k |a_k|^2 |e_k\rangle\langle e_k|, \end{aligned} \tag{26}$$

the eigenvalues of ϱ_A are the squared Schmidt coeff. of $|\psi_{AB}\rangle$, i.e. "Schmidt-rank" $(\varrho_{AB}) = \mathrm{rank}\,(\varrho_A) = \mathrm{rank}\,(\varrho_B)$.

Example: A two qubit state: $|\xi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ with corresponding density operator

$$\varrho_{AB} = |\xi\rangle\langle\xi| = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}. \tag{27}$$

This leads to

$$\begin{aligned}\varrho_A =& \mathrm{tr}_B(\varrho_{AB}) = \sum_i (\mathbb{1}_A \otimes \langle i_B|)\, \varrho_{AB}\, (\mathbb{1}_A \otimes |i_B\rangle) \\ =& (\mathbb{1}_A \otimes \langle 0|)\, \varrho_{AB}\, (\mathbb{1}_A \otimes |0\rangle) + (\mathbb{1}_A \otimes \langle 1|)\, \varrho_{AB}\, (\mathbb{1}_A \otimes |1\rangle) \\ =& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \\ & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ =& \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ =& \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbb{1}. \end{aligned} \tag{28}$$

The reduced state of subsystem A is the maximally mixed state, i.e. the state $|\xi\rangle$ has Schmidt rank two and it is maximally entangled (see next section).

Quantifying pure state entanglement

Def. (von Neumann entropy): The von Neumann entropy of a quantum system with density matrix ϱ is

$$S(\varrho) = -\mathrm{tr}(\varrho \log \varrho)\,. \tag{29}$$

(base 2 for log)
Note: The logarithm of ϱ is defined via the spectral decomposition:

$$\log \varrho = \sum_i (\log \lambda_i)\, |i\rangle\langle i|, \tag{30}$$

where λ_i are the eigenvalues and $|i\rangle$ are the eigenvectors of ϱ. Therefore

$$S(\varrho) = -\sum_k \langle k| \left(\sum_i \lambda_i \log \lambda_i |i\rangle\langle i| \right) |k\rangle = -\sum_k \lambda_k \log \lambda_k. \tag{31}$$

The von Neumann entropy is quantum analogon of the classical Shannon entropy.

An entanglement measure for pure states:

$$\begin{aligned} E(|\psi_{AB}\rangle = S(\varrho_A) = S(\varrho_B) \\ = -\sum_k |a_k|^2 \log |a_k|^2, \end{aligned} \tag{32}$$

is the entropy of entanglement, with a_k being the Schmidt coefficients.

Examples:

$|\psi_{AB}\rangle = |00\rangle \rightarrow \varrho_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow S(\varrho_A) = 0$ being separable,

$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \varrho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow S(\varrho_A) = 1$ being maximal entangled.

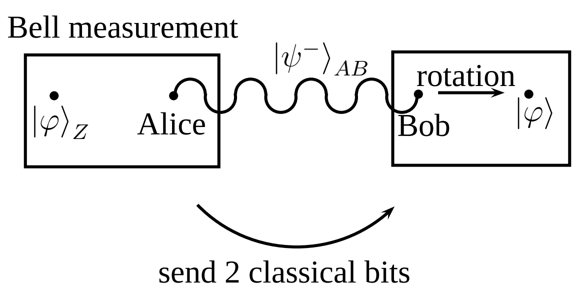
4 Entanglement as a resource: Quantum teleportation

Teleportation (Oxford dictionary): *"Teleportation is the apparently instantaneous transportation of persons etc. across space by advanced technological means."*

[C. Bennett *et al.* , Phys. Rev. Lett. **70**, 1895 (1993)]
Alice and Bob share $|\psi^-\rangle_{\mathrm{AB}}$ (a singlet). Alice has an additional qubit in an unknown state

$$|\varphi\rangle_{\mathrm{Z}} = \alpha\,|0\rangle_{\mathrm{Z}} + \beta\,|1\rangle_{\mathrm{Z}} \tag{33}$$

and wants to transfer it to Bob:



Write the total state $|\psi_{\mathrm{total}}\rangle = |\varphi\rangle_{\mathrm{Z}} \otimes |\psi^-\rangle_{\mathrm{AB}}$ in the Bell basis for system ZA:

$$\begin{aligned} |00\rangle &= \frac{1}{\sqrt{2}}[|\phi^+\rangle + |\phi^-\rangle], \\ |01\rangle &= \frac{1}{\sqrt{2}}[|\psi^+\rangle + |\psi^-\rangle], \\ |10\rangle &= \frac{1}{\sqrt{2}}[|\psi^+\rangle - |\psi^-\rangle], \\ |11\rangle &= \frac{1}{\sqrt{2}}[|\phi^+\rangle - |\phi^-\rangle]. \end{aligned} \tag{34}$$

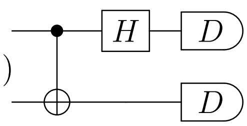
We have

$$\begin{aligned}
|\psi_{\text{total}}\rangle &= |\varphi\rangle_{\text{Z}} \otimes |\psi^-\rangle_{\text{AB}} \\
&= [\alpha|0\rangle + \beta|1\rangle]_{\text{Z}} \otimes \frac{1}{\sqrt{2}}[|0\rangle_{\text{A}}|1\rangle_{\text{B}} - |1\rangle_{\text{A}}|0\rangle_{\text{B}}] \\
&= \frac{1}{2}\big(\alpha[|\phi^+\rangle + |\phi^-\rangle]_{\text{ZA}}|1\rangle_{\text{B}} - \alpha([|\psi^+\rangle + |\psi^-\rangle]_{\text{ZA}}|0\rangle_{\text{B}} \\
&\quad +\beta[|\psi^+\rangle - |\psi^-\rangle]_{\text{ZA}}|1\rangle_{\text{B}} - \beta[|\phi^+\rangle - |\phi^-\rangle]_{\text{ZA}}|0\rangle_{\text{B}}\big) \\
&= \frac{1}{2}\big(|\phi^+\rangle_{\text{ZA}}[\alpha|1\rangle - \beta|0\rangle]_{\text{B}} + |\phi^-\rangle_{\text{ZA}}[\alpha|1\rangle + \beta|0\rangle]_{\text{B}} \\
&\quad + |\psi^+\rangle_{\text{ZA}}[-\alpha|0\rangle + \beta|1\rangle]_{\text{B}} + |\psi^-\rangle_{\text{ZA}}[-\alpha|0\rangle - \beta|1\rangle]_{\text{B}}\big)\,.
\end{aligned} \tag{35}$$

The teleportation protocol:

1. Alice does a Bell measurement on her 2 qubits A and Z, e.g. by unitary operations and single qubit measurements:

   $(|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle)$

   $(0, 1, 0, 1)$

   $(0, 0, 1, 1)$

   The measurement operators are

   $$\{P_\mu\} = \{|\phi^+\rangle\langle\phi^+|, |\phi^-\rangle\langle\phi^-|, |\psi^+\rangle\langle\psi^+|, |\psi^-\rangle\langle\psi^-|\}\,. \tag{36}$$

   The state after the measurement is given by

   $$|\psi_\mu^{\text{total}}\rangle = \frac{P_\mu \otimes \mathbb{1}_B |\psi^{\text{total}}\rangle}{\sqrt{\langle\psi^{\text{total}}| P_\mu \otimes \mathbb{1}_B |\psi^{\text{total}}\rangle}}. \tag{37}$$

2. Alice tells Bob (classical channel) the outcome of her measurement.

3. Bob rotates his quantum state by a unitary operation (one of the Pauli operators):

Alice's outcome	$\lvert\phi^+\rangle$	$\lvert\phi^-\rangle$	$\lvert\psi^+\rangle$	$\lvert\psi^-\rangle$
Bob's rotation	$i\sigma_y$	σ_x	$-\sigma_z$	$-\mathbb{1}$

   Remember:

   $$\begin{aligned}
   \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0| \\
   \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \\
   \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|\,.
   \end{aligned} \tag{38}$$

Remarks:

- "Quantum teleportation" transports no matter, only information.
- The state transfer is not instantaneous, classical information has to be transmitted from Alice to Bob.
- With classical information, Bob can reconstruct Alice's state.
- Alice and Bob need $|\psi^-\rangle_{\mathrm{AB}}$ as a ressource.

Successful experiments in Innsbruck [D. Bouwmeester *et al.* , Nature **390**, 575 (1997)] and Rome [D. Boschi *et al.* , Phys. Rev. Lett. **80**, 1121 (1998)].

5 Superdense coding

Transmit two bits of information by transmitting only one qubit [C. Bennett, S. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992)]. Again Alice and Bob share an entangled state $|\psi^-\rangle_{\mathrm{AB}}$.

The protocol:

1. Alice performs a local rotation, one out of the following four:

$$\begin{aligned} \mathbb{1}_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} \left|\psi^-\right\rangle_{\mathrm{AB}} &= \left|\psi^-\right\rangle_{\mathrm{AB}} \\ (\sigma_x)_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} \left|\psi^-\right\rangle_{\mathrm{AB}} &= -\left|\phi^-\right\rangle_{\mathrm{AB}} \\ (\sigma_y)_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} \left|\psi^-\right\rangle_{\mathrm{AB}} &= i\left|\phi^+\right\rangle_{\mathrm{AB}} \\ (\sigma_z)_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{B}} \left|\psi^-\right\rangle_{\mathrm{AB}} &= \left|\psi^+\right\rangle_{\mathrm{AB}} . \end{aligned} \tag{39}$$

2. Alice sends her qubit to Bob.
3. Bob does a Bell measurement on both qubits and finds one of the four outcomes (two bits of information were transmitted).